RATED EXTREMAL PRINCIPLES FOR FINITE AND INFINITE SYSTEMS*

BORIS S. MORDUKHOVICH† and HUNG M. PHAN‡

Dedicated to Juan Enrique Martinez-Legaz in honor of his 60th birthday

Abstract. In this paper we introduce new notions of local extremality for finite and infinite systems of closed sets and establish the corresponding extremal principles for them called here rated extremal principles. These developments are in the core geometric theory of variational analysis. We present their applications to calculus and optimality conditions for problems with infinitely many constraints.

Key words. Variational analysis, extremal principles, generalized normals, calculus rules, infinite intersections, semi-infinite and infinite optimization, necessary optimality conditions

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1 Introduction

Modern variational analysis is based on variational principles and techniques applied to optimization-related and equilibrium problems as well as to a broad spectrum of problems, which may not be of a variational nature; see the books [1, 8, 9, 13] for more discussions and references. In this vein, extremal principles have been well recognize as fundamental geometric tools of variational analysis and its applications that can be treated as far-going variational extensions of convex separation theorems to systems of nonconvex sets. We refer the reader to the two-volume monograph [8, 9] and the bibliographies therein for various developments and applications of the extremal principles in both finite and infinite dimensions.

To the best of our knowledge, extremal principles have been previously developed only for finite systems of sets. On the other, there is a strong demand in various areas (e.g., in semi-infinite optimization) for their counterparts involving infinite, particularly countable, set systems.

The first attempt to deal with infinite systems of sets was undertaken in our recent papers [10, 11], where certain tangential extremal principles were established for countable set systems and then were applied therein to problems of semi-infinite programming and multiobjective optimization. At the same time, the tangential extremal principles developed and applied in [10, 11] concern the so-called tangential extremality (and only in finite dimensions) and do not reduce to the conventional extremal principles of [8] for finite systems of sets even in simple frameworks.

In this paper we develop new *rated extremal principles* for both finite and infinite systems of closed sets in finite-dimensional and infinite-dimensional spaces. Besides being applied to conventional local extremal points of finite set systems and reducing to the known results for them, the rated extremal principles provide enhanced information in the case of finitely many sets while open new lines of development for countable set systems. The results obtained in this way allow us, in particular, to derive intersection rules for generalized normals of infinite intersections of closed sets, which imply in turn new necessary optimality conditions for mathematical programs with countable constraints in finite and infinite dimensions.

The rest of the paper is organized as follows. In Section 2 we briefly discussed preliminaries from variational analysis and generalized differentiations used in the sequel. In Section 3 we introduce the notion of rated extremality and derive exact and approximate versions of the rated extremal principles for systems of finite sets in finite-dimensional and infinite-dimensional spaces. Section 4 is devoted to rated extremal principles for infinite/countable systems of closed sets in Banach spaces. Finally, Section 5 provides applications of the rated extremal principles to calculus of generalized normals to infinite set intersections, which

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†Department of Mathematics, Wayne State University, Detroit, MI 48202, USA. Email: boris@math.wayne.edu.

‡Department of Mathematics, Wayne State University, Detroit, MI 48202, USA. Email: pmhung@wayne.edu.

implies necessary optimality conditions for optimization problems with countable geometric constraints.

Our notation is basically standard in variational analysis; see, e.g., [8, 13]. Recall that $B(\bar{x}, r)$ stands for a closed ball centered at $\bar{x}$ with radius $r > 0$, that $I\!B$ and $I\!B^*$ are the closed unit ball of the space in question and its dual, respectively, and that $I\!N := \{1, 2, \ldots\}$. Given a set-valued mapping $F\colon X \rightrightarrows X^*$ between a Banach space X and its topological dual X^*, we denote by

$$\mathop{\rm Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{ x^* \in X^* \in Y \Big|\ \exists \text{ sequences } \ x_k \to \bar{x} \ \text{ and } \ x_k^* \xrightarrow{w^*} x^* \ \text{ as } \ k \to \infty \ \text{ such that } \ x_k^* \in F(x_k) \text{ for all } \ k \in I\!N \Big\} \tag{1.1}$$

the *sequential Painlevé-Kuratowski outer limit* of F at $\bar{x}$, where w^* signifies the weak* topology of X^*.

2 Preliminaries from Variational Analysis

In this section we briefly overview some basic tools of variational analysis and generalized differentiation that are widely used in what follows; see the books [1, 8, 13, 14] for more details and references. Unless otherwise stated, all the spaces under consideration are Banach with the norm $\|\cdot\|$ and the canonical pairing $\langle\cdot,\cdot\rangle$ between the space in question and its topological dual.

Let $\emptyset$ be a nonempty subset of a space X. Given $\varepsilon \ge 0$, the *set of ε-normals* to $\emptyset$ at $\bar{x}$ is given by

$$\widehat{N}_\varepsilon(\bar{x}; \emptyset) := \left\{ x^* \in X^* \Big|\ \limsup_{x \xrightarrow{\emptyset} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \le \varepsilon \right\} \tag{2.1}$$

with $\widehat{N}_\varepsilon(\bar{x}; \emptyset) := \emptyset$ if $\bar{x} \notin \emptyset$. When $\varepsilon = 0$, the set (2.1) is denoted by $\widehat{N}(\bar{x}; \emptyset) := \widehat{N}_0(\bar{x}; \emptyset)$ and is called the *Fréchet normal cone* (or *prenormal/regular normal cone*) to $\emptyset$ at $\bar{x}$. The *Mordukhovich/basic/limiting normal cone* to $\emptyset$ at a point $\bar{x} \in \emptyset$ is defined by

$$N(\bar{x}; \emptyset) := \mathop{\rm Lim\,sup}_{\substack{x \to \bar{x} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x; \emptyset) \tag{2.2}$$

via the sequential outer limit Painlevé-Kuratowski outer limit (1.1) of ε-normals (2.1) as $x \to \bar{x}$ and $\varepsilon \downarrow 0$. If the space X is Asplund (i.e., each of its separable subspace has a separable dual that holds, in particular, when is reflexive) and the set $\emptyset$ is locally closed around $\bar{x}$, we can equivalently put $\varepsilon_k = 0$ in (2.2); see [8] for more details. If $X = \mathbb{R}^n$, the basic normal cone (2.2) can be equivalently described as

$$N(\bar{x}; \emptyset) = \mathop{\rm Lim\,sup}_{x\to\bar{x}} \Big\{ \text{cone}\, \big[x - \Pi(x; \emptyset) \big] \Big\} \tag{2.3}$$

via the Euclidian projector $\Pi(x; \emptyset) := \{ w \in \emptyset |\ \|x - w\| = \text{dist}\,(x; \emptyset)\}$ of $x \in \mathbb{R}^n$ onto $\emptyset$, which was the original definition in [7]. In the above formula (2.3) the symbol cone A stands for the cone generated by a nonempty set A and is defined by

$$\text{cone}\, A := \bigcup_{\lambda \ge 0} \lambda A.$$

Given an extended-real-valued function $\varphi\colon X \to \overline{\mathbb{R}} := (-\infty, \infty]$, recall that the *Fréchet/regular subdifferential* of φ at $\bar{x}$ with $\varphi(\bar{x}) < \infty$ is defined by

$$\widehat{\partial}\varphi(\bar{x}) := \Big\{ x^* \in X^* \Big|\ \liminf_{x\to\bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \ge 0 \Big\}. \tag{2.4}$$

It is easy to see that $\widehat{N}(\bar{x}; \emptyset) = \widehat{\partial}\delta(\bar{x}; \emptyset)$ for the indicator function $\delta(\cdot; \emptyset)$ of $\emptyset$ defined by $\delta(x; \emptyset) := 0$ when $x \in \emptyset$ and $\delta(x; \emptyset) = \infty$ otherwise. Furthermore, we obviously have the following nonsmooth version of the Fermat stationary rule:

$$0 \in \widehat{\partial}\varphi(\bar{x}) \ \text{ if } \ \bar{x} \ \text{ is a local minimizer of } \ \varphi. \tag{2.5}$$

A major motivation for our work is to develop and apply extremal principles of variational analysis the first version of which was formulated in [6] for finitely many sets via ε-normals (2.1); see [8, Chapter 2] for more details and discussions. Recall [8, Definition 2.5] that a set system $\{\emptyset_1,\ldots,\emptyset_m\}$, $m\ge 2$, satisfies the *approximate extremal principle* at $\bar{x}\in\cap_{i=1}^m\emptyset_i$ if for every $\varepsilon>0$ there are $x_i\in\emptyset_i\cap(\bar{x}+\varepsilon I\!B)$ and $x_i^*\in\widehat{N}(x_i;\emptyset_i)+\varepsilon I\!B^*$, $i=1,\ldots,m$, such that

$$x_1^*+\ldots+x_m^*=0\quad\text{and}\quad\|x_1^*\|^2+\ldots+\|x_m^*\|^2=1. \tag{2.6}$$

If the dual vectors x_i^* can be taken from the limiting normal cone $N(\bar{x};\emptyset_i)$, then we say that the system $\{\emptyset_1,\ldots,\emptyset_m\}$ satisfies the *exact extremal principle* at $\bar{x}$.

Efficient conditions ensuring the fulfillment of both approximate and exact versions of the extremal principle can be found in [8, Chapter 2] and the references therein. Roughly speaking, the approximate extremal principle in terms of Fréchet normals holds for locally extremal points of any closed subsets in Asplund spaces ([8, Theorem 2.20]) while the exact extremal principle requires additional sequential normal compactness assumptions that are automatic in finite dimensions; see [8, Theorem 2.22].

Recall [6, 8] that a point $\bar{x}\in\cap_{i=1}^m\emptyset_i$ is *locally extremal* for the system $\{\emptyset_1,\ldots,\emptyset_m\}$ if there are sequences $\{a_{ik}\}\subset X$, $i=1,\ldots,m$, and a neighborhood U of $\bar{x}$ such that $a_{ik}\to 0$ as $k\to\infty$ and

$$\bigcap_{i=1}^m\Big(\emptyset_i-a_{ik}\Big)\cap U=\emptyset\ \text{ for all large }\ k\in I\!N. \tag{2.7}$$

As shown in [8], this extremality notion for sets encompasses standard notions of local optimality for various optimization-related and equilibrium problems as well as for set systems arising in proving calculus rules and other frameworks of variational analysis.

3 Rated Extremality of Finite Systems of Sets

In this section we introduce a new notion of *rated extremality* for finite systems of sets, which essentially broader the previous notion (2.7) of local extremality. We show nevertheless that both exact and approximate versions of the extremal principle hold for this rated extremality under the same assumptions as in [8] for locally extremal points. Let us start with the definition of rated extremal points. For simplicity we drop the word "local" for rated extremal points in what follows.

Definition 3.1 (Rated extremal points of finite set systems)**.** *Let $\emptyset_1,\ldots,\emptyset_m$ as $m\ge 2$ be nonempty subsets of X, and let $\bar{x}$ be a common point of these sets. We say that $\bar{x}$ is a (local)* RATED EXTREMAL POINT *of rank α, $0\le\alpha<1$, of the set system $\{\emptyset_1,\ldots,\emptyset_m\}$ if there are $\gamma>0$ and sequences $\{a_{ik}\}\subset X$, $i=1,\ldots,m$, such that $r_k:=\max_i\|a_{ik}\|\to 0$ as $k\to\infty$ and*

$$\bigcap_{i=1}^m\big(\emptyset_i-a_{ik}\big)\cap B(\bar{x},\gamma r_k^\alpha)=\emptyset\quad\text{for all large }\ k\in I\!N. \tag{3.1}$$

In this case we say that $\{\emptyset_1,\ldots,\emptyset_m\}$ is a RATED EXTREMAL SYSTEM *at $\bar{x}$.*

The case of local extremality (2.7) obviously corresponds to (3.1) with rate $\alpha=0$. The next example shows that there are rated extremal points for systems of two simple sets in $\mathbb{R}^2$, which are not locally extremal in the conventional sense of (2.7).

Example 3.2 (Rated extremality versus local extremality)**.** Consider the sets

$$\emptyset_1:=\big\{(x_1,x_2)\in\mathbb{R}^2\big|\ x_2-x_1^2\le 0\big\}\ \text{ and }\ \emptyset_2:=\big\{(x_1,x_2)\in\mathbb{R}^2\big|\ -x_2-x_1^2\le 0\big\}.$$

Then it is easy to check that $(\bar{x}_1,\bar{x}_2)=(0,0)\in\emptyset_1\cap\emptyset_2$ is a rated extremal point of rank $\alpha=\frac{1}{2}$ for the system $\{\emptyset_1,\emptyset_2\}$ but not a local extremal point of this system.

Prior to proceeding with the main results of this section, we briefly discuss relationships between the rated extremality and the *tangential extremality* of set systems introduced in [10]. Let $\{\varnothing_i, i = 1, \ldots, m\}$, $m \geq 2$, be a system of sets with $\bar{x} \in \cap_{i=1}^m \varnothing_i$, and let $\Lambda := \{\Lambda_i(\bar{x}), i = 1, \ldots, m\}$ be an approximating system of cones. Recall that $\bar{x}$ is a Λ-tangential local extremal point of $\{\varnothing_i, i = 1, \ldots, m\}$ if the system of cones $\{\Lambda_i(\bar{x}), i = 1, \ldots, m\}$ is extremal at the origin in the sense that there are $a_1, \ldots, a_m \in X$ such that

$$\bigcap_{i=1}^{m} (\varnothing_i - a_i) = \emptyset.$$

We refer the reader to [10, 11] for more discussion on the tangential extremality and its applications.

The next proposition result and the subsequent example reveal relationships between the rated extremality and tangential extremality of set systems.

Proposition 3.3 (Relationships between rated and tangential extremality of finite systems of sets)**.** *Let $\{\varnothing_1, \ldots, \varnothing_m\}$ as $m \geq 2$ be a Λ-tangential extremal system of sets at $\bar{x}$. Assume that there are real numbers $C > 0$, $p \in (0, 1)$ and a neighborhood U of $\bar{x}$ such that*

$$\text{dist}\,(x - \bar{x}; \Lambda_i) \leq C\|x - \bar{x}\|^{1+p} \ \text{ for all } \ x \in \varnothing_i \cap U \ \text{ and } \ i = 1, \ldots, m. \tag{3.2}$$

Then $\{\varnothing_1, \ldots, \varnothing_m\}$ is a rated extremal system at $\bar{x}$.

Proof. Since the general case of $m \geq 2$ can be derived by induction, it suffices to justify the result in the case of $m = 2$. Let $\{\Lambda_1, \Lambda_2\}$ be an extremal system of approximation cones and find by definition elements $a_1, a_2 \in X$ such that

$$(\Lambda_1 - a_1) \cap (\Lambda_2 - a_2) = \emptyset.$$

Without loss of generality, assume that $a_1 = -a_2 =: a$. Take $\alpha \in (0, 1)$ with $\beta := \alpha(1 + p) > 1$ and show that for all small $t > 0$ we have

$$(\varnothing_1 - ta) \cap (\varnothing_2 + ta) \cap B(\bar{x}, \|ta\|^\alpha) = \emptyset. \tag{3.3}$$

Suppose by contradiction that there exists

$$x \in (\varnothing_1 - ta) \cap (\varnothing_2 + ta) \cap B(\bar{x}, \|ta\|^\alpha). \tag{3.4}$$

That implies by using condition (3.2) that

$$\begin{aligned}
\text{dist}\,(x - \bar{x}; \Lambda_1 - ta) &= \text{dist}\,(x + ta - \bar{x}; \Lambda_1) \leq C\|x + ta - \bar{x}\|^{1+p},\\
\text{dist}\,(x - \bar{x}; \Lambda_2 + ta) &= \text{dist}\,(x - ta - \bar{x}; \Lambda_2) \leq C\|x - ta - \bar{x}\|^{1+p}.
\end{aligned}$$

Thus we have for some constant $\widetilde{C}$ that

$$\|x + ta - \bar{x}\|^{1+p} \leq \widetilde{C} \max\big\{\|x - \bar{x}\|, \|ta\|\big\}^{1+p} \leq \widetilde{C} \max\big\{\|ta\|^\beta, \|ta\|^{1+p}\big\} = o(\|ta\|) \ \text{ as } \ t \downarrow 0$$

and similarly $\|x - ta - \bar{x}\|^{1+p} = o(\|ta\|)$. Put then $d := \text{dist}\,(\Lambda_1 - a, \Lambda_2 + a) > 0$ and observe due the conic structures of Λ_1 and Λ_2 that

$$td = \text{dist}\,(\Lambda_1 - ta; \Lambda_2 + ta) > 0$$

for all $t > 0$ sufficiently small. Combining all the above gives us

$$td = \text{dist}\,(\Lambda_1 - ta; \Lambda_2 + ta) \leq \text{dist}\,(x - \bar{x}; \Lambda_1 - ta) + \text{dist}\,(x - \bar{x}; \Lambda_2 + ta) = o(\|ta\|),$$

which is a contradiction. Thus $\{\varnothing_1, \varnothing_2, \bar{x}\}$ is a rated extremal system at $\bar{x}$ with rank α chosen above. This completes the proof of the proposition. $\square$

One of the most important special cases of tangential extremality is the so-called *contingent extremality* when the approximating cones to $\varnothing_i$ are given by the Bouligand-Severi contingent cones to this sets; see [10, 11], where this case of tangential extremality was primarily studied and applied. The following example (of two parts) shows that the notions of rated extremality and contingent extremality are independent from each other in a simple setting of two sets in $\mathbb{R}^2$.

Example 3.4 (Independence of rated and contingent extremality)**.** Let $X = \mathbb{R}^2$, and let $\bar{x} = (0,0)$.

(i) Consider two closed sets in $\mathbb{R}^2$ given by

$$\O_1 := \operatorname{epi} f \ \text{ and } \ \O_2 := \mathbb{R} \times \mathbb{R}_- \setminus \operatorname{int} \O_1,$$

where $f(x) := x \sin \frac{1}{x}$ for $x \in \mathbb{R}$ with $f(0) := 0$. It is easy to see that the contingent cones to $\O_1$ and $\O_2$ at $\bar{x}$ are computed by

$$\Lambda_1 = \operatorname{epi}(-|\cdot|) \ \text{ and } \ \Lambda_2 = \mathbb{R} \times \mathbb{R}_-.$$

We can check that the set system $\{\O_1, \O_2\}$ is locally extremal at $\bar{x}$, and hence $\bar{x}$ is a rated extremal point of this system of sets with rank $\alpha = 0$. On the other hand, the contingent extremality is obviously violated for $\{\O_1, \O_2\}$ at $\bar{x}$ as follows from the above computations of Λ_1 and Λ_2.

(ii) Now we define two closed sets in $\mathbb{R}^2$ by

$$\O_1 := \mathbb{R} \times \mathbb{R}_- \ \text{ and } \ \O_2 := \operatorname{epi} \ \text{ with } \ f(x) := -x^{1 + \frac{1}{\ln^2 |x|}} \ \text{ for } \ x \neq 0 \ \text{ and } \ f(0) := 0.$$

The contingent cones to $\O_1$ and $\O_2$ at $\bar{x}$ are easily computed by $\Lambda_1 = \mathbb{R} \times \mathbb{R}_-$ and $\Lambda_2 = \mathbb{R} \times \mathbb{R}_+$. We can check that $\bar{x}$ is not a rated extremal point of $\{\O_1, \O_2\}$ whenever $\alpha \in [0,1)$, while the contingent extremality obviously holds for this system at $\bar{x}$.

The next theorem justifies the fulfillment of the exact extremal principle for any rated extremal point of a finite system of closed sets in $\mathbb{R}^n$. It extends the extremal principle of [8, Theorem 2.8] obtained for local extremal points, i.e., when $\alpha = 0$ in Definition 3.1.

Theorem 3.5 (Exact extremal principle for rated extremal systems of sets in finite dimensions)**.** *Let $\bar{x}$ be a rated extremal point of rank $\alpha \in [0,1)$ for the system of sets $\{\O_1, \ldots, \O_m\}$ as $m \geq 2$ in $\mathbb{R}^n$. Assume that all the sets $\O_i$ are locally closed around $\bar{x}$. Then the exact extremal principle holds for $\{\O_1, \ldots, \O_m\}$ at $\bar{x}$, i.e, there are $x_i^* \in N(\bar{x}; \O_i)$ for $i = 1, \ldots, m$ satisfying the relationships in* (2.6).

Proof. Given a rated extremal point $\bar{x}$ of the system $\{\O_1, \ldots, \O_m\}$, take numbers $\alpha \in [0,1)$ and $\gamma > 0$ as well as sequences $\{a_{ik}\}$ and $\{r_k\}$ from Definition 3.1. Consider the following unconstrained minimization problem for any fixed $k \in I\!N$:

$$\text{minimize } \ d_k(x) := \left[\sum_{i=1}^m \operatorname{dist}^2\big(x + a_{ik}; \O_i\big) \right]^{\frac{1}{2}} + \frac{\sqrt{m}}{\gamma^{\frac{1}{\alpha}}} \|x - \bar{x}\|^{\frac{1}{\alpha}}, \ x \in \mathbb{R}^n. \tag{3.5}$$

Since the function d_k is continuous and its level sets are bounded, there exists an optimal solution x_k to (3.5) by the classical Weierstrass theorem. We obviously have the relationships

$$d_k(x_k) \leq d_k(\bar{x}) = \left[\sum_{i=1}^m \operatorname{dist}^2\big(\bar{x} + a_{ik}; \O_i\big) \right]^{\frac{1}{2}} \leq \left[\sum_{i=1}^m \|a_{ik}\|^2 \right]^{\frac{1}{2}} \leq r_k \sqrt{m},$$

which readily imply the estimate

$$\frac{\sqrt{m}}{\gamma^{\frac{1}{\alpha}}} \|x_k - \bar{x}\|^{\frac{1}{\alpha}} \leq r_k \sqrt{m}, \ \text{ i.e., } \ \|x_k - \bar{x}\| \leq \gamma r_k^\alpha.$$

Taking the latter into account, we get

$$\nu_k := \left[\sum_{i=1}^m \operatorname{dist}^2\big(x_k + a_{ik}; \O_i\big) \right]^{\frac{1}{2}} > 0,$$

since the opposite statement $\nu_k = 0$ contradicts the rated extremality of $\bar{x}$. Furthermore, the optimality of x_k in (3.5) and choice of $\{a_{ik}\}$ give us the relationships

$$d_k(x_k) = \nu_k + \frac{\sqrt{m}}{\gamma^{\frac{1}{\alpha}}} \|x_k - \bar{x}\|^{\frac{1}{\alpha}} \leq \left[\sum_{i=1}^m \|a_{ik}\|^2 \right]^{\frac{1}{2}} \downarrow 0 \ \text{ as } \ k \to \infty,$$

which ensure in turn that $x_k \to \bar{x}$ and $\nu_k \downarrow 0$ as $k \to \infty$.

We now arbitrarily pick $w_{ik} \in \Pi(x_k + a_{ik}; Ø_i)$ for $i = 1, \ldots, m$ in the closed set $Ø_i$ and for each $k \in I\!N$ consider the problem:

$$\text{minimize} \quad \rho_k(x) := \left[\sum_{i=1}^{m} \|x + a_{ik} - w_{ik}\|^2 \right]^{\frac{1}{2}} + \frac{\sqrt{m}}{\gamma^{\frac{1}{\alpha}}} \|x - \bar{x}\|^{\frac{1}{\alpha}}, \quad x \in \mathbb{R}^n, \tag{3.6}$$

which obviously has the same optimal solution x_k as for (3.5). Since $\nu_k > 0$ and the norm $\|\cdot\|$ is Euclidian, the function $\rho_k(\cdot)$ in (3.6) is continuously differentiable around x_k. Thus applying the classical Fermat rule to the *smooth* unconstrained minimization problem (3.6), we get

$$\nabla \rho_k(x_k) = \sum_{i=1}^{m} x_{ik}^* + C\|x_k - \bar{x}\|^{\frac{1-2\alpha}{\alpha}} (x_k - \bar{x}) = 0 \quad \text{for some constant } C,$$

where $x_{ik}^* := (x_k + a_{ik} - w_{ik})/\nu_k$ for $i = 1, \ldots, m$ with

$$\|x_{1k}^*\|^2 + \ldots + \|x_{mk}^*\|^2 = 1.$$

Observe that $\|x_k - \bar{x}\|^{\frac{1-2\alpha}{\alpha}} (x_k - \bar{x}) = \|x_k - \bar{x}\|^{\frac{1-\alpha}{\alpha}} \dfrac{x_k - \bar{x}}{\|x_k - \bar{x}\|} \to 0$ as $x_k \to \bar{x}$. Due to the compactness of the unit sphere in $\mathbb{R}^n$, we find $x_i^* \in \mathbb{R}^n$ as $i = 1, \ldots, m$ such that $x_{ik}^* \to x_i^*$ as $k \to \infty$ without relabeling. It follows from the equivalent description (2.3) of the limiting normal cone that $x_i^* \in N(\bar{x}; Ø_i)$ for all $i = 1, \ldots, m$. Moreover, we get from the constructions above that

$$\|x_1^*\|^2 + \ldots + \|x_m^*\|^2 = 1 \quad \text{and} \quad x_1^* + \ldots + x_m^* = 0.$$

This gives all the conclusions of the exact extremal principle and completes the proof of the theorem. $\square$

The next example shows that the exact extremal principle is violated if we take $\alpha = 1$ in Definition 3.1.

Example 3.6 (Violating the exact extremal principle for rated extremal points of rank $\alpha = 1$)**.** Define two closed sets in $\mathbb{R}^2$ by

$$Ø_1 := \text{epi}\,(-\|\cdot\|) \quad \text{and} \quad Ø_2 := \mathbb{R} \times \mathbb{R}_-.$$

Taking any $a_k \downarrow 0$, we see that

$$\big(Ø_1 + (0, a_k)\big) \cap \big(Ø_1 - (0, a_k)\big) \cap B(\bar{x}, a_k/2) = \emptyset,$$

i.e., $\bar{x} = (0, 0)$ is a rated extremal point of $\{Ø_1, Ø_2\}$ of rank $\alpha = 1$. However, it is easy to check that the relationships of the exact extremal principle do not hold for this system at $\bar{x}$.

Observe that Example 3.6 shows that the relationships of the approximate extremal principle are also violated when $\alpha = 1$. However, for rated extremal systems of rank $\alpha \in [0, 1)$ the approximate extremal principle holds in general infinite-dimensional settings. Let us proceed with justifying this statement extending the corresponding results of [8] obtained for the rank $\alpha = 0$ in Definition 3.1.

Theorem 3.7 (Approximate extremal principle for rated extremal systems in Fréchet smooth spaces)**.** *Let X be a Banach space admitting an equivalent norm Fréchet differentiable off the origin, and let $\bar{x}$ be a rated extremal point of rank $\alpha \in [0, 1)$ for a system of sets $Ø_1, \ldots, Ø_m$ locally closed around $\bar{x}$. Then the approximate extremal principle holds for $\{Ø_1, \ldots, Ø_m\}$ at $\bar{x}$.*

Proof. Choose an equivalent norm $\|\cdot\|$ on X differentiable off the origin and consider first the case of $m = 2$ in the theorem. Let $\bar{x} \in Ø_1 \cap Ø_2$ be a rated extremal point of rank $\alpha \in [0, 1)$ with $\gamma > 0$ taken from Definition 3.1. Denote $r := \max\{\|a_1\|, \|a_2\|\}$ and for any $\varepsilon > 0$ find a_1, a_2 such that

$$r^{1-\alpha} \le \min\left\{\frac{\gamma}{2}, \frac{\varepsilon}{(2\gamma)^{(1-\alpha)/\alpha}}\right\} \quad \text{and} \quad \big(Ø_1 - a_1\big) \cap \big(Ø_2 - a_2\big) \cap B\big(\bar{x}, \gamma r^{\alpha}\big) = \emptyset.$$

We also select a constant $C>0$ with $(\frac{2}{C})^\alpha=\frac{\gamma}{2}$ and denote $\beta:=\frac{1}{\alpha}>1$. Define the function

$$\varphi(z):=\|(x_1-a_1)-(x_2-a_2)\| \quad \text{for} \quad z=(x_1,x_2)\in X\times X \tag{3.7}$$

with the product norm $\|z\|:=(\|x_1\|^2+\|x_2\|^2)^{1/2}$ on $X\times X$, which is Fréchet differentiable off the origin under this property of the norm on X. Next fix $z_0=(\bar{x},\bar{x})$ and define the set

$$W(z_0):=\big\{z\in \Omega_1\times\Omega_2\big|\varphi(z)+C\|z-z_0\|^\beta\le\varphi(z_0)\big\}\,, \tag{3.8}$$

which is obviously nonempty and closed. For each $z=(x_1,x_2)\in W(z_0)$ we have $i=1,2$:

$$C\|x_i-\bar{x}\|^\beta\le C\|z-\bar{z}\|^\beta\le\varphi(z_0)=\|-a_1+a_2\|\le 2r,\quad i=1,2,$$

which implies that $\|x_i-\bar{x}\|\le\big(\frac{2}{C}\big)^{\frac{1}{\beta}}r^{\frac{1}{\beta}}=\big(\frac{2}{C}\big)^{\alpha}r^\alpha=\frac{\gamma}{2}r^\alpha$ and thus

$$W(z_0)\subset B(\bar{x},\gamma r^\alpha)\times B(\bar{x},\gamma r^\alpha)\subset B\big(\bar{x},\tfrac{1}{2}\varepsilon^{\frac{\alpha}{1-\alpha}}\big)\times B\big(\bar{x},\tfrac{1}{2}\varepsilon^{\frac{\alpha}{1-\alpha}}\big).$$

It follows from Definition 3.1 and constructions (3.7) and (3.8) that $\varphi(z)>0$ for all $z\in W(x_0)$. Indeed, assuming on the contrary that $\varphi(z)=0$ for some $z=(x_1,x_2)\in W(x_0)$ gives us

$$\|x_1-a_1-\bar{x}\|\le\|x_1-\bar{x}\|+\|a_1\|\le\tfrac{\gamma}{2}r^\alpha+r=\big(\tfrac{\gamma}{2}+r^{1-\alpha}\big)r^\alpha\le\gamma r^\alpha$$

and thus $x_1-a_1=x_2-a_2\in\big(\Omega_1-a_1\big)\cap\big(\Omega_2-a_2\big)\cap B(\bar{x},\gamma r^\alpha)\ne\emptyset$, a contradiction.

Hence φ is Fréchet differentiable at any point $z\in W(z_0)$. Pick any $z_1\in\Omega_1\times\Omega_2$ satisfying

$$\varphi(z_1)+C\|z_1-z_0\|^\beta\le\inf_{W(z_0)}\Big\{\varphi(z)+C\|z-z_0\|^\beta\Big\}+\frac{r}{2}$$

and define further the nonempty and closed set

$$W(z_1):=\left\{z\in\Omega_1\times\Omega_2\Big|\ \varphi(z)+C\|z-z_0\|^\beta+C\frac{\|z-z_1\|^\beta}{2}\le\varphi(z_1)+C\|z_1-z_0\|^\beta\right\}.$$

Arguing inductively, suppose we have chosen z_k and constructed $W(z_k)$, then pick $z_{k+1}\in W(z_k)$ such that

$$\varphi(z_{k+1})+C\sum_{i=0}^{k}\frac{\|z_{k+1}-z_i\|^\beta}{2^i}\le\inf_{W(z_k)}\Big\{\varphi(z)+C\sum_{i=0}^{k}\frac{\|z-z_i\|^\beta}{2^i}\Big\}+\frac{r}{2^{2k+1}}$$

and construct the subsequent nonempty and closed set

$$W(z_{k+1}):=\left\{z\in\Omega_1\times\Omega_2\Big|\ \varphi(z)+C\sum_{i=0}^{k+1}\frac{\|z-z_i\|^\beta}{2^i}\le\varphi(z_{k+1})+C\sum_{i=0}^{k}\frac{\|z_{k+1}-z_i\|^\beta}{2^i}\right\}.$$

It is easy to see that the sequence $\{W(z_k)\}\subset\Omega_1\times\Omega_2$ is nested. Let us check that

$$\operatorname{diam}W(z_{k+1}):=\sup\big\{\|z-w\|\,\big|\,z,w\in W(z_{k+1})\big\}\to 0\ \text{ as }\ k\to\infty. \tag{3.9}$$

Indeed, for each $z\in W(z_{k+1})$ and $k\in\mathbb{N}$ we have

$$\begin{aligned}C\frac{\|z-z_{k+1}\|^\beta}{2^{k+1}}&\le\varphi(z_{k+1})+C\sum_{i=0}^{k}\frac{\|z_{k+1}-z_i\|^\beta}{2^i}-\left(\varphi(z)+C\sum_{i=0}^{k}\frac{\|z-z_i\|^\beta}{2^i}\right)\\&\le\varphi(z_{k+1})+C\sum_{i=0}^{k}\frac{\|z_{k+1}-z_i\|^\beta}{2^i}-\inf_{W(z_k)}\Big\{\varphi(z)+C\sum_{i=0}^{k}\frac{\|z-z_i\|^\beta}{2^i}\Big\}\le\frac{r}{2^{2k+1}},\end{aligned}$$

which implies that $\operatorname{diam} W(z_{k+1}) \le 2\left(\dfrac{r}{C2^k}\right)^{\frac{1}{\beta}}$ and thus justifies (3.9). Due to the completeness of X the classical Cantor theorem ensures the existence of $\bar z = (\bar x_1, \bar x_2) \in W(z_0)$ such that $\bigcap\limits_{k=0}^{\infty} W(z_k) = \{\bar z\}$ with $z_k \to \bar z$ as $k \to \infty$. Now we show that $\bar z$ is a minimum point of the function

$$\phi(z) := \varphi(z) + C\sum_{i=0}^{\infty}\frac{\|z-z_i\|^\beta}{2^i} \tag{3.10}$$

over the set $\varnothing_1 \times \varnothing_2$. To proceed, take any $\bar z \neq z \in \varnothing_1 \times \varnothing_2$ and observe that $z \notin W(z_k)$ for all $k \in I\!N$ sufficiently large while $\bar z \in W(z_k)$. This yields the estimates

$$\phi(z) \ge \varphi(z) + C\sum_{i=0}^{k}\frac{\|z-z_i\|^\beta}{2^i} \ge \varphi(z_k) + C\sum_{i=0}^{k-1}\frac{\|z_k-z_i\|^\beta}{2^i} \ge \varphi(\bar z) + C\sum_{i=0}^{k}\frac{\|\bar z-z_i\|^\beta}{2^i}$$

and hence justifies the claimed inequality $\phi(z) \ge \phi(\bar z)$ by letting $k \to \infty$.

We get therefore that the function $\phi(z) + \delta(z; \varnothing_1 \times \varnothing_2)$ attains at $\bar z$ its minimum on the whole space $X \times X$. The generalized Fermat rule (2.5) gives us the inclusion $0 \in \widehat\partial\big(\phi(z)+\delta(z;\varnothing_1\times\varnothing_2)\big)$. Since $\varphi(\bar z) > 0$ and the norm $\|\cdot\|^\beta$ is smooth, the function ϕ in (3.10) is Fréchet differentiable at $\bar z$. Applying the sum rule from [8, Proposition 1.107], the Fréchet subdifferential formula for the indicator function, and the product formula for Fréchet normal cone (2.1) from [8, Proposition 1.2], we get

$$-\nabla\phi(\bar z) = -(u_1^*, u_2^*) \in \widehat N(\bar z; \varnothing_1 \times \varnothing_2) = \widehat N(\bar x_1; \varnothing_1) \times \widehat N(\bar x_2; \varnothing_2),$$

where the dual elements u_i^*, $i = 1, 2$, are computed by

$$u_1^* = x^* + \sum_{j=0}^{\infty} w_{1j}^*\frac{\|\bar x_1 - x_{1j}\|^{\beta-1}}{2^j} \quad\text{and}\quad u_2^* = -x^* + \sum_{j=0}^{\infty} w_{2j}^*\frac{\|\bar x_2 - x_{2j}\|^{\beta-1}}{2^j}$$

with $z_j = (x_{1j}, x_{2j})$, $x^* = \nabla\big(\|\cdot\|\big)\big((\bar x_1 - a_1) - (\bar x_2 - a_2)\big)$, and

$$w_{ij}^* = \begin{cases}\nabla(\|\cdot\|)(\bar x_i - x_{ij}) & \text{if } \bar x_i - x_{ij} \neq 0,\\ 0 & \text{otherwise.}\end{cases}$$

for $i = 1, 2$ and $j = 0, 1, \ldots$ due to the construction of the function ϕ in (3.10). Observing further that $\|x^*\| = 1$ and that $\bar z, z_i \in W(z_0)$ gives us

$$\|\bar x_i - x_{ij}\| \le \varepsilon^{\frac{1-\alpha}{\alpha}} = \varepsilon^{\frac{1}{\beta-1}},$$

which implies the estimates $\|\bar x_i - x_{ij}\|^{\beta-1} \le \varepsilon$ and

$$\sum_{j=0}^{\infty}\|w_{ij}^*\|\frac{\|\bar x_i - x_{ij}\|^{\beta-1}}{2^j} \le 2\varepsilon, \quad i = 1, 2.$$

Setting finally $x_1^* := -x^*/2$, $x_2^* := x^*/2$, and $x_i := \bar x_i$ for $i = 1, 2$, we arrive at the relationships

$$\begin{aligned}&x_i^* \in \widehat N(x_i; \varnothing_i) + \varepsilon B^*\ ,\ x_i \in B(\bar x, \varepsilon) \ \text{ for } \ i = 1, 2,\\ &\|x_1^*\| + \|x_2^*\| = 1, \ \text{ and } \ x_1^* + x_2^* = 0,\end{aligned}$$

which show that the approximate extremal principle holds for rated extremal points of two sets.

Consider now the general case of $m > 2$ sets. Observe that if $\bar x$ as a rated extremal point of the system $\{\varnothing_1, \ldots, \varnothing_m\}$ with some rank $\alpha \in [0, 1)$, then the point $\bar z := (\bar x, \ldots, \bar x) \in X^{n-1}$ is a local rated extremal point of the same rank for the system of two sets

$$\Theta_1 := \varnothing_1 \times \ldots \times \varnothing_{n-1} \ \text{ and } \ \Theta_2 := \big\{(x, \ldots, x) \in X^{n-1}\big| x \in \varnothing_m\big\}. \tag{3.11}$$

To justify this, take numbers $\alpha \in [0,1)$ and $\gamma > 0$ and the sequences $(a_{1k}, \ldots, a_{mk})$ from Definition 3.1 for m sets and check that

$$\Big(\Theta_1 - (a_{1k}, \ldots, a_{n-1,k})\Big) \cap \Big(\Theta_2 - (a_{nk}, \ldots, a_{nk})\Big) \cap B\big((\bar{x}, \ldots, \bar{x}); \gamma r_k^{\alpha}\big) = \emptyset \tag{3.12}$$

with $r_k := \max\{\|a_{1k}\|, \ldots, \|a_{nk}\|\}$. Indeed, the violation of (3.12) means that there are $(x_1, \ldots, x_{n-1}) \in \emptyset_1 \times \ldots \times \emptyset_{n-1}$ and $x_m \in \emptyset_m$ satisfying

$$x_1 - a_{1k} = \ldots = x_{m-1} - a_{m-1,k} = x_m - a_{mk} \in B(\bar{x}, \gamma r_k^{\alpha}),$$

which clearly contradicts the rated extremality of $\bar{x}$ with rank α for the system $\{\emptyset_1, \ldots, \emptyset_m\}$. Applying finally the relationships of the approximate extremal principle to the system of two sets in (3.11) and taking into account the structures of these sets as well as the aforementioned product formula for Fréchet normals, we complete the proof of the theorem. $\square$

The next theorem elevates the fulfillment of the approximate extremal principle for rated extremal points from Fréchet smooth to Asplund spaces by using the method of *separable reduction*; see [3, 8].

Theorem 3.8 (Approximate extremal principle for rated extremal systems in Asplund spaces)**.** *Let X be an Asplund space, and let $\bar{x}$ be a rated extremal point of rank $\alpha \in [0,1)$ for a system of sets $\emptyset_1, \ldots, \emptyset_m$ locally closed around $\bar{x}$. Then the approximate extremal principle holds for $\{\emptyset_1, \ldots, \emptyset_m\}$ at $\bar{x}$.*

Proof. Taking a rated extremal point $\bar{x}$ for the system $\{\emptyset_1, \ldots, \emptyset_m\}$ of rank $\alpha \in [0,1)$, find a number $\gamma > 0$ and sequences $\{a_{ik}\}$, $i = 1, \ldots, m$, from Definition 3.1. Consider a separable subspace Y_0 of the Asplund space X defined by

$$Y_0 := \operatorname{span}\big\{\bar{x}, a_{ik} \,\big|\, i = 1, \ldots, m,\ k \in I\!N\big\}.$$

Pick now a closed and separable subspace $Y \subset X$ with $Y \supset Y_0$ and observe that $\bar{x}$ is a rated extremal point of rank α for the system $\{\emptyset_1 \cap Y, \ldots, \emptyset_m \cap Y\}$. Indeed, we have

$$\begin{aligned} &\Big((\emptyset_1 \cap Y) - a_{1k}\Big) \cap \ldots \cap \Big((\emptyset_m \cap Y) - a_{mk}\Big) \cap B_Y(\bar{x}; \gamma r_k^{\alpha}) \\ &\subset \Big(\emptyset_1 - a_{1k}\Big) \cap \ldots \cap \Big(\emptyset_m - a_{mk}\Big) \cap B_X(\bar{x}; \gamma r_k^{\alpha}) = \emptyset, \end{aligned}$$

where $r_k := \max\{\|a_{1k}\|, \ldots, \|a_{mk}\|\}$, and where B_X and B_Y are the closed unit balls in the space X and Y, respectively. The rest of the proof follows the one in [8, Theorem 2.20] by taking into account that Y admits an equivalent Fréchet differentiable norm off the origin. $\square$

We conclude this section with deriving the exact extremal principle for rated extremal systems of rank $\alpha \in [0,1)$ in Asplund spaces extending the corresponding result of [8, Theorem 2.22] obtained for $\alpha = 0$.

Recall that a set $\emptyset \subset X$ is *sequentially normally compact* (SNC) at $\bar{x} \in \emptyset$ if for any sequence $\{(x_k, x_k^*)\}_{k \in I\!N} \subset \emptyset \times X^*$ we have the implication

$$\big[x_k \to \bar{x},\ x_k^* \stackrel{w^*}{\to} 0 \ \text{ with } \ x_k^* \in \widehat{N}(x_k; \emptyset),\ k \in I\!N\big] \Longrightarrow \|x_k^*\| \to 0 \ \text{ as } \ k \to \infty. \tag{3.13}$$

Besides the obvious validity of this property in finite-dimensional spaces, it holds also in broad infinite-dimensional settings; see, in particular, [8, Subsection 1.2.5] and SNC calculus rules established in [8, Section 3.3] in the framework of Asplund spaces.

Theorem 3.9 (Exact extremal principle for rated extremal systems in Asplund spaces)**.** *Let X be an Asplund space, and let $\bar{x}$ be a rated extremal point of rank $\alpha \in [0,1)$ for a system of sets $\emptyset_1, \ldots, \emptyset_m$ locally closed around $\bar{x}$. Assume that all but one of the sets $\emptyset_i$, $i = 1, \ldots, m$, are SNC at $\bar{x}$. Then the exact extremal principle holds for $\{\emptyset_1, \ldots, \emptyset_m\}$ at $\bar{x}$.*

Proof. Follows the lines in the proof of [8, Theorem 2.22] by passing to the limit in the relationships of the rated approximate extremal principle obtained in Theorem 3.8. $\square$

4 Rated Extremal Principles for Infinite Set Systems

This section concerns new notions of rated extremality and deriving rated extremal principles for infinite systems of closed sets. The main results are obtained in the framework of Asplund spaces.

Let us start with introducing a notion of rated extremality for arbitrary (may be infinite and not even countable) systems of sets in general Banach spaces. We say that $R(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a *rate function* if there is a real number M such that

$$rR(r) \le M \quad \text{and} \quad \lim_{r\downarrow 0} R(r) = \infty. \tag{4.1}$$

In what follow we denote by $|I|$ the cardinality (number of elements) of a finite set I.

Definition 4.1 (Rated extremality for infinite systems of sets). *Let $\{\Omega_i\}_{i\in T}$ be a system of closed subsets of X indexed by an arbitrary set T, and let $\bar{x} \in \bigcap_{t\in T}\Omega_i$. Given a rate function $R(\cdot)$, we say that $\bar{x}$ is an R-*RATED EXTREMAL POINT *of the system $\{\Omega_i\}_{i\in T}$ if there exist sequences $\{a_{ik}\} \subset X$, $i \in T$ and $k \in \mathbb{N}$, with $r_k := \sup_{i\in T}\|a_{ik}\| \to 0$ as $k \to \infty$ such that whenever $k \in \mathbb{N}$ there is a finite index subset $I_k \subset T$ of cardinality $|I_k|^{3/2} = o(R_k)$ with $R_k := R(r_k)$ satisfying*

$$\bigcap_{i\in I_k}\big(\Omega_i - a_{ik}\big) \cap B\big(\bar{x}; r_kR_k\big) = \emptyset \quad \textit{for all large } k. \tag{4.2}$$

*In this case we say that $\{\Omega_i\}_{i\in T}$ is an R-*RATED EXTREMAL SYSTEM *at $\bar{x}$.*

It is easy to see that a finite rated extremal system of sets from Definition 3.1 is a particular case of Definition 4.1. Indeed, suppose that $\bar{x}$ is a rated extremal point of rank $\alpha \in [0, 1)$ for a finite set system $\{\Omega_1, \ldots, \Omega_m\}$, i.e., condition (3.1) is satisfied. Defining $R(r) := \frac{\gamma}{r^{1-\alpha}}$, we have that $rR(r) \to 0$ and $R(r) \to \infty$ as $r \to 0$; thus $R(\cdot)$ is a rate function while condition (4.2) is satisfied.

Let us discuss some specific features of the rated extremality in Definition 4.1 for the case of infinite systems. For simplicity we denote $R = R(r)$ in what follows if no confusion arises.

Remark 4.2 (Growth condition in rated extremality). Observe that, although $\{\Omega_i\}_{i\in T}$ is an infinite system in Definition 4.1, the rated extremality therein involves only *finitely many* sets for each given accuracy $\varepsilon > 0$. The imposed requirement $|I|^{3/2} = o(R)$ guarantees that $|I|^{3/2}$ *grows slower* than R, which is very crucial in our proof of the extremal principle below. In other words, the number of sets involved must not be *too large*; otherwise the result is trivial. We prove in Theorem 4.6 that the rate $|I|^{3/2} = o(R)$ ensures the validity of the rated extremal principle, where the number r measures *how far* the sets are shifted.

Define next extremality conditions for infinite systems of sets, which we are going to justify as an appropriate extremal principle in what follows. These conditions are of the approximate extremal principle type expressed in terms of of Fréchet normals at points nearby the reference one.

Definition 4.3 (Rated extremality conditions for infinite systems). *Let $\{\Omega_i\}_{i\in T}$ be a system of nonempty subsets of X indexed by an arbitrary set T, and let $\bar{x} \in \bigcap_{t\in T}\Omega_i$. We say that the set system $\{\Omega_i\}_{i\in T}$ satisfies the* RATED EXTREMAL PRINCIPLE *at $\bar{x}$ if for any $\varepsilon > 0$ there exist a number $r \in (0, \varepsilon)$, an finite index subset $I \subset T$ with cardinality $|I|r < \varepsilon$, points $x_i \in \Omega_i \cap B(\bar{x}, \varepsilon)$, and dual elements $x_i^* \in \widehat{N}(x_i; \Omega_i) + r\mathbb{B}^*$ for $i \in I$ such that*

$$\sum_{i\in I} x_i^* = 0 \quad \textit{and} \quad \sum_{i\in I}\|x_i^*\|^2 = 1. \tag{4.3}$$

Observe that when a system consists of finitely many sets $\{\Omega_1, \ldots, \Omega_m\}$ with $|I| = m$, we put the other sets equal to the whole space X and reduce Definition 4.1 in this case to the conventional conditions of the approximate extremal principle for finite systems of sets; see Section 2.

Now we address the nontriviality issue for the introduced version of the extremal principle for infinite set systems. It is appropriate to say (roughly speaking) that a version of the extremal principle is *trivial*

if all the information is obtained from only one set of the system while the other sets contribute nothing; i.e., if $y_i^* = 0 \in \widehat{N}(x_i; \O_i)$ for all but one index i. This issue was first addressed in [10], where it has been shown that a "natural" extension of the approximate extremal principle for countable systems is trivial.

The next proposition justifies the nontriviality of the rated extremal principle for infinite set systems proposed in Definition 4.3.

Proposition 4.4 (Nontriviality of rated extremality conditions for infinite systems)**.** *Let $\{\O_i\}_{i\in T}$ be a system of set satisfying the extremality conditions of Definition 4.3 at some point $\bar{x} \in \bigcap_{t\in T} \O_i$. Then the rated extremal principle defined by these conditions is nontrivial.*

Proof. Suppose on the contrary that the rated extremal principle of Definition 4.3 is trivial, i.e., there is $i_0 \in T$ (say $i_0 = 1$) and $y_i^* \in X^*$ as $i \in T$ such that

$$x_i^* \in y_i^* + r I\!\!B^* \subset \widehat{N}(x_i; \O_i) + r I\!\!B^* \ \text{ for all } \ i \in I,$$

$$\sum_{i\in I} x_i^* = 0, \ \sum_{i\in I} \|x_i^*\|^2 = 1, \ \text{ and } \ y_i^* = 0 \ \text{ whenever } \ i \in I \setminus \{1\}$$

in the notation of Definition 4.1. It follows that $\|x_i^*\| \le r$ for all $i \in I \setminus \{1\}$ implying that

$$\Big\| y_1^* + \sum_{i\ne 1} x_i^* \Big\| \le r \ \text{ and } \ \|y_1^*\| \le |I| r.$$

Thus we arrive at the relationships

$$\sum_{i\in I} \|x_i^*\|^2 < (\|y_1^*\| + r)^2 + \sum_{i\ne 1} r^2 \le |I|^2 r^2 + 2|I| r^2 + r^2 + (|I| - 1) r^2 < C\varepsilon^2 \downarrow 0$$

as $\varepsilon \downarrow 0$, a contradiction. This justifies the nontriviality of the rated extremal principle. $\square$

Observe further that the extremal principle of Definition 4.3 may be trivial is the rate condition $|I| r < \varepsilon$ is not imposed. The following example describes a general setting when this happens.

Example 4.5 (The rate condition is essential for nontriviality)**.** Assume that the condition $|I| r < \varepsilon$ is violated in the framework of Definition 4.3. Fix $\nu > 0$, suppose that $I = \{1, \ldots, N\}$ with $Nr > \nu$, pick some $u^* \in \widehat{N}(x_1; \O_1)$ with the norm $\|u^*\| = \nu$, and define the dual elements

$$\begin{aligned} x_1^* &:= u^* - \frac{u^*}{N} \in \widehat{N}(x_1; \O_1) + r I\!\!B^*, \\ x_i^* &:= 0 - \frac{u^*}{N} \in \widehat{N}(x_i; \O_i) + r I\!\!B^* \ \text{ for all } \ i = 2, \ldots, N. \end{aligned}$$

Then we have the relationships

$$x_1^* + \ldots + x_N^* = 0 \ \text{ and } \ \|x_1^*\|^2 + \ldots + \|x_N^*\|^2 > \frac{\nu^2}{4},$$

which imply the triviality of the rated extremal principle by rescaling.

Now we are ready to derive the main result of this section, which justifies the validity of the rated extremal principle for rated extremal points of infinite systems of closed sets in Asplund spaces.

Theorem 4.6 (Rated extremal principle for infinite systems)**.** *Let $\{\O_i\}_{i\in T}$ be a system of closed sets in an Asplund space X, and let $\bar{x}$ be a rated extremal point of this system. Then the rated extremality conditions of Definition 4.3 are satisfied for $\{\O_i\}_{i\in T}$ at $\bar{x}$.*

Proof. Given $\varepsilon > 0$, take $r = \sup_i \|a_i\|$ sufficiently small and pick the corresponding index subset $I = \{1, \ldots, N\}$ with $N^{3/2} = o(R)$ from Definition 4.1. Consider the product space X^N with the norm of $z = (x_1, \ldots, x_N) \in X^N$ given by

$$\|z\| := (\|x_1\|^2 + \ldots + \|x_N\|^2)^{\frac{1}{2}}$$

and define a function $\varphi\colon X^N \to \mathbb{R}$ by

$$\varphi(z) := \left(\sum_{i=2}^{N} \|(x_1 - a_1) - (x_i - a_i)\|^2 \right)^{\frac{1}{2}}. \tag{4.4}$$

To proceed, denote $\bar{z} := (\bar{x}, \bar{x}, \ldots, \bar{x}) \in \Omega_1 \times \ldots \times \Omega_N$ and form the set

$$W := \Big(\Omega_1 \times \ldots \times \Omega_N\Big) \cap \Big(B\big(\bar{x}, (R-1)r\big) \times \ldots \times B\big(\bar{x}, (R-1)r\big)\Big), \tag{4.5}$$

which is nonempty and closed. We conclude that $\varphi(z) > 0$ for all $z \in W$. Indeed, suppose on the contrary that $\varphi(z) = 0$ for some $z = (x_1, \ldots, x_N) \in W$ and get by the estimates $\|x_1 - a_1 - \bar{x}\| \le \|x_1 - \bar{x}\| + \|a_1\| \le (R-1)r + r = Rr$ the relationships

$$x_1 - a_1 = \ldots = x_N - a_N \in \bigcap_{i=1}^{N} (\Omega_i - a_i) \cap B(\bar{x}, Rr) \ne \emptyset,$$

which contradict the extremality condition (4.2). Observe further that

$$\varphi(\bar{z}) = \left(\sum_{i=2}^{N} \|a_1 - a_i\|^2 \right)^{\frac{1}{2}} < 2r\sqrt{N} \le \inf_{z \in W} \varphi(z) + 2rN^{\frac{1}{2}}.$$

Now we apply Ekeland's variational principle (see, e.g., [8, Theorem 2.26]) with the parameters

$$\varepsilon := 2rN^{\frac{1}{2}} \quad \text{and} \quad \lambda := rR^{\frac{1}{2}}N^{\frac{3}{4}}$$

to the lower semicontinuous and bounded from below function $\varphi(z) + \delta(z; W)$ on X^N and find in this way $z_0 \in W$ such that $\|z_0 - \bar{z}\| \le \lambda$ and that z_0 minimizes the perturbed function

$$\varphi(z) + \beta\|z - z_0\| + \delta(z; W) \quad \text{on} \quad z \in X^N \quad \text{with} \quad \beta := \frac{\varepsilon}{\lambda} = \frac{2}{R^{\frac{1}{2}}N^{\frac{1}{4}}}. \tag{4.6}$$

By the imposed growth condition $N^{\frac{3}{2}} = o(R)$ as $r \downarrow 0$ we have

$$\begin{aligned}
&\varepsilon = 2rN^{\frac{1}{2}} = r \cdot o(R^{\frac{1}{3}}) \le r \cdot o\Big(\frac{1}{r}\Big)^{\frac{1}{3}} \le r \cdot o\Big(\frac{1}{r}\Big) \to 0,\\
&\frac{\lambda}{Rr} = \frac{rR^{\frac{1}{2}}N^{\frac{3}{4}}}{Rr} = \frac{N^{\frac{3}{4}}}{R^{\frac{1}{2}}} \to 0,\\
&N\beta = \frac{2N}{R^{\frac{1}{2}}N^{\frac{1}{4}}} = \frac{2N^{\frac{3}{4}}}{R^{\frac{1}{2}}} = 2\Big(\frac{N^{\frac{3}{2}}}{R}\Big)^{\frac{1}{2}} \to 0 \quad \text{as} \quad r \downarrow 0.
\end{aligned}$$

Thus $\lambda = o(Rr)$ and $\beta \downarrow 0$ as $r \downarrow 0$ for the quantity β defined in (4.6). Taking into account that the function $\varphi(\cdot) + \beta\|\cdot - z_0\|$ is obviously Lipschitz continuous around $\bar{z}$, we apply to this sum the subdifferential fuzzy sum rule from [8, Lemma 2.32]. This allows us to find, for any given number $\eta > 0$, elements $z_1 = (y_1, \ldots, y_N) \in z_0 + \eta I\!B$ and $z_2 = (x_1, \ldots, x_N) \in z_0 + \eta I\!B$ such that

$$\big|\varphi(z_1) + \beta\|z_1 - z_0\| - \varphi(z_0)\big| \le \eta, \; z_2 \in W, \quad \text{and} \tag{4.7}$$

$$0 \in \widehat{\partial}\Big(\varphi(\cdot) + \beta\|\cdot - z_0\|\Big)(z_1) + \widehat{N}(z_2; W) + \eta I\!B^*. \tag{4.8}$$

Our next step is to explore formula (4.8). Since $\varphi(z_0) > 0$, we choose

$$\eta \le \min\Big\{\beta, \lambda, \frac{\varphi(z_0)}{2(1+\beta)}\Big\}.$$

Then it follows from (4.7) that

$$|\varphi(z_1) - \varphi(z_0)| \le (1+\beta)\eta \le (1+\beta)\frac{\varphi(z_0)}{2(1+\beta)} = \frac{\varphi(z_0)}{2},$$

which implies that $\varphi(z_1) =: \alpha > 0$. It is easy to see that the function $\varphi(\cdot)$ in (4.4) is convex. Applying the Moreau-Rockafellar theorem of convex analysis gives us

$$\widehat{\partial}\Big(\varphi(\cdot) + \beta\|\cdot - z_0\|\Big)(z_1) = \widehat{\partial}\varphi(z_1) + \beta I\!B^*, \tag{4.9}$$

where the Fréchet subdifferentials on both sides of (4.9) reduce to the classical subdifferential of convex functions. By the structure of φ in (4.4) and that of z_1 we have

$$\varphi(z_1) = \left(\sum_{i=2}^{N}\|(y_1 - a_1) - (y_i - a_i)\|^2\right)^{\frac{1}{2}}.$$

Denote further $\xi_i := y_1 - a_1 - y_i + a_i$ for $i = 2, \ldots, N$ and observe that $\alpha = \varphi(z_1) = \Big(\sum_{i=2}^{N}\|\xi_i\|^2\Big)^{\frac{1}{2}}$. Since the square root function is smooth at nonzero point, we apply the chain rule of convex analysis to derive that any element $(y_1^*, \ldots, y_N^*) \in \widehat{\partial}\varphi(z_1)$ has the representation

$$y_i^* = \begin{cases} -\dfrac{u_i^*}{\alpha}\cdot\|\xi_i\| & \text{if } \xi_i \neq 0, \\ 0 & \text{if } \xi_i = 0, \end{cases} \qquad i = 2, \ldots, N,$$

and $y_1^* = -y_2^* - y_3^* - \ldots - y_N^*$, where $u_i^* \in \widehat{\partial}\|\cdot\|(\xi_i)$ is a subgradient of the norm function calculated at the nonzero point ξ_i; hence $\|u_i^*\| = 1$. This yields that

$$\|y_2^*\|^2 + \ldots + \|y_N^*\|^2 = 1 \ \text{ and } \ \|y_1^*\|^2 + \ldots + \|y_N^*\|^2 \geq 1.$$

On the other hand, we have the estimates

$$\|z_2 - \bar{z}\| \leq |z_2 - z_0\| + \|z_0 - \bar{z}\| \leq \eta + \lambda \leq 2\lambda = o(Rr)$$

for $z_2 = (x_1, \ldots, x_N)$ and hence $\|x_i - \bar{x}\| < \|z_2 - \bar{z}\| = o(Rr)$ for $i = 1, \ldots, N$. The latter ensures that each component x_i lies in the interior of the ball $B(\bar{x}, (R-1)r)$. Furthermore, it follows from the structure of W in (4.5) and the product formula for Fréchet normals that

$$\widehat{N}(z_2; W) = \widehat{N}\big(z_2; \O_1 \times \ldots \times \O_N\big) = \widehat{N}(x_1; \O_1) \times \ldots \times \widehat{N}(x_N; \O_N),$$

which implies by combining with (4.8) and (4.9) the existence of $(y_1^*, \ldots, y_N^*) \in \widehat{\partial}\varphi(z_1)$ satisfying

$$\begin{aligned} &0 \in y_i^* + \widehat{N}(x_i; \O_i) + 2\beta I\!B^*, \ \ \|x_i - \bar{x}\| < 2\lambda \to 0 \ \text{ as } \ r \downarrow 0, \\ &y_1^* + \ldots + y_N^* = 0, \ \text{ and } \ \|y_1^*\|^2 + \ldots + \|y_N^*\|^2 > 1. \end{aligned}$$

Finally, replace y_i^* by $-y_i^*$ and get from the above that

$$\begin{aligned} &y_i^* \in \widehat{N}(x_i; \O_i) + 2\beta I\!B^*, \ \ \|x_i - \bar{x}\| < 2\lambda \to 0, \\ &\text{for } \ i = 1, \ldots, N, \ \ N\beta \to 0 \ \text{ as } \ r \downarrow 0, \\ &y_1^* + \ldots + y_N^* = 0, \ \text{ and } \ \|y_1^*\|^2 + \ldots + \|y_N^*\|^2 \geq 1, \end{aligned}$$

which gives all the relationships of the rated extremal principle and completes the proof of the theorem. □

From the proof above we can distill some quantitative estimates for the elements involved in the relationships of the rated extremal principle.

Remark 4.7 (Quantitative estimates in the rated extremal principle)**.** The proof of Theorem 4.6 essentially uses the growth assumptions $N^{3/2} = o(R)$ and $R \leq \frac{M}{r}$ on rated extremal points. Observe in fact that the given proof allows us to make the following *quantitative conclusions*: For any $\varepsilon > 0$ there exist a number $r \in (0, \varepsilon)$, an index subset $I = \{j_1, \ldots, j_N\}$ with $N^{3/2} = o(R(r))$, and elements

$$y_i^* \in \widehat{N}(x_i; \O_i) \ \text{ with } \ \|x_i - \bar{x}\| \leq 2rR^{\frac{1}{2}}N^{\frac{3}{4}} \ \text{ for all } \ i \in I$$

satisfying the relationships

$$\|y_{j_1}^* + \ldots + y_{j_N}^*\| \le 2N\beta = \frac{4N^{\frac{3}{4}}}{R^{\frac{1}{2}}} \text{ and } \|y_{j_1}^*\|^2 + \ldots + \|y_{j_N}^*\|^2 \ge 1.$$

Similar but somewhat different quantitative statement can be also made: For any rated extremal point $\bar{x}$ of the system $\{\Omega_i\}_{i\in T}$ with a rate function $R(r) = O(r)$ there is a constant $C > 0$ such that whenever $\varepsilon > 0$ there exist a number $r \in (0, \varepsilon)$, an index subset $I = \{j_1, \ldots, j_N\}$ with $N^{3/2} = o(\frac{1}{r})$, and elements

$$y_i^* \in \widehat{N}(x_i; \Omega_i) \text{ with } \|x_i - \bar{x}\| \le C\sqrt{rN^{\frac{3}{2}}} \text{ for all } i \in I$$

satisfying the estimates

$$\|y_{j_1}^* + \ldots + y_{j_N}^*\| \le C\sqrt{rN^{\frac{3}{2}}} \text{ and } \|y_{j_1}^*\|^2 + \ldots + \|y_{j_N}^*\|^2 \ge 1.$$

In the last part of this section we introduce and study a certain notion of *perturbed extremality* for arbitrary (finite or infinite) set systems and compare it, in particular, with the notion of linear subextremality known for systems of two sets. Given two sets $\Omega_1, \Omega_2 \subset X$, the number

$$\vartheta(\Omega_1, \Omega_2) := \sup\left\{\nu \ge 0 \middle|\, \nu I\!B \subset \Omega_1 - \Omega_2\right\}$$

is known as the *measure of overlapping* for these sets [5]. We say that the system $\{\Omega_1, \Omega_2\}$ is *linear subextremal* [9, Subsection 5.4.1] around $\bar{x}$ if

$$\vartheta_{lin}(\Omega_1, \Omega_2, \bar{x}) := \liminf_{\substack{x_1 \xrightarrow{\Omega_1} \bar{x},\, x_2 \xrightarrow{\Omega_2} \bar{x} \\ r \downarrow 0}} \frac{\vartheta\Big([\Omega_1 - x_1] \cap rI\!B, [\Omega_2 - x_2] \cap rI\!B\Big)}{r} = 0, \tag{4.10}$$

which is called "weak stationarity" in [5]; see [5, 9] for more discussions and references. It is proved in [5] and [9, Theorem 5.88] that the linear subextremality of a closed set system $\{\Omega_1, \Omega_2\}$ around $\bar{x}$ is equivalent, in the Asplund space setting, to the validity of the approximate extremal principle for $\{\Omega_1, \Omega_2\}$ at $\bar{x}$.

Our goal in what follows is to define a perturbed version of rated extremality, which is applied to infinite set systems while extends linear subextremality for systems of two sets as well. Given an R-rated extremal system of sets $\{\Omega_i\}_{i\in T}$ from Definition 4.1, we get that for any $\varepsilon > 0$ there are $r = \sup\|a_i\|$, $R = R(r)$, and $I \subset T$ satisfying

$$\bigcap_{i\in I}\big(\Omega_i - \bar{x} - a_i\big) \cap (rR)I\!B = \emptyset. \tag{4.11}$$

Let us now perturb (4.11) by replacing $\bar{x}$ with some $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and arrive at the following construction.

Definition 4.8 (Perturbed extremal systems)**.** *Let $\{\Omega_i\}_{i\in T}$ be a system of nonempty sets in X, and let $\bar{x} \in \bigcap_{i\in T}\Omega_i$. We say that $\bar{x}$ is R-*PERTURBED EXTREMAL POINT *of $\{\Omega_i, i \in T\}$ if for any $\varepsilon > 0$ there exist $r = \sup_{i\in I}\|a_i\| < \varepsilon$, $I \subset T$ with $|I|^{3/2} = o(R)$, and $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ as $i \in I$ such that*

$$\bigcap_{i\in I}\big(\Omega_i - x_i - a_i\big) \cap (rR)I\!B = \emptyset. \tag{4.12}$$

*In this case we say that $\{\Omega_i\}_{i\in T}$ is an R-*PERTURBED EXTREMAL SYSTEM *at $\bar{x}$.*

The next proposition establishes a connection between linear subextremality and perturbed extremality for systems of two sets $\{\Omega_1, \Omega_2\}$.

Proposition 4.9 (Perturbed extremality from linear subextremality)**.** *Let a set system $\{\Omega_1, \Omega_2, \bar{x}\}$ be linearly subextremal around $\bar{x}$. Then it is an R-perturbed extremal system at this point.*

Proof. Employing the definition of linear subextremality, for any $\varepsilon > 0$ sufficiently small we find $x_i \in \O_i \cap B_\varepsilon(\bar{x})$ and $r' < \varepsilon$ such that

$$\vartheta\big([\O_1 - x_1] \cap r' I\!\!B, [\O_2 - x_2] \cap r' I\!\!B\big) < r'\varepsilon.$$

This implies the existence of a vector $a \in X$ satisfying $\|a\| \le r'\varepsilon$ and

$$a \notin \Big([\O_1 - x_1] \cap r' I\!\!B\Big) - \Big([\O_2 - x_2] \cap r' I\!\!B\Big),$$

which ensures in turn that

$$\Big([\O_1 - x_1] \cap r' I\!\!B - \frac{a}{2}\Big) \cap \Big([\O_2 - x_2] \cap r' I\!\!B + \frac{a}{2}\Big) = \emptyset. \tag{4.13}$$

Let us show that the latter implies the fulfillment of

$$\Big[\O_1 - x_1 - \frac{a}{2}\Big] \cap \Big[\O_2 - x_2 + \frac{a}{2}\Big] \cap \frac{r'}{2} I\!\!B = \emptyset. \tag{4.14}$$

Indeed, suppose that (4.14) does not hold and pick $\xi \in X$ from the left-hand side set in (4.14). Since $\xi + \frac{a}{2} \in \O_1 - x_1$ and $\|\xi\| \le \frac{r'}{2}$, we have

$$\left\|\xi_{\frac{a}{2}}\right\| \le \frac{r'}{2} + \frac{r'\varepsilon}{2} \le \frac{r'}{2} + \frac{r'}{2} = r'$$

and consequently $\xi \in [\O_1 - x_1] \cap r' I\!\!B - \dfrac{a}{2}$. Similarly we get $\xi \in [\O_2 - x_2] \cap r' I\!\!B - \dfrac{a}{2}$. This clearly contradicts (4.13) and thus justifies the claimed relationship (4.14).

By setting $r := \dfrac{\|a\|}{2}$, out remaining task is to construct a continuous function $: \mathbb{R}_+ \to \mathbb{R}_+$ such that $R(r) \to \infty$ as $r \downarrow 0$ and that for each $\varepsilon > 0$ there is $r < \varepsilon$ satisfying

$$\Big[\O_1 - x_1 - \frac{a}{2}\Big] \cap \Big[\O_2 - x_2 + \frac{a}{2}\Big] \cap (rR) I\!\!B = \emptyset.$$

We first construct such a function along a sequence $r_k \downarrow 0$ as $k \to \infty$. Picking $\varepsilon_k \downarrow 0$, find $r'_k < \varepsilon_k$ and select $a_k \in X$ with $\|a_k\| \le r'_k \varepsilon_k$ such that the sequence of $\|a_k\|$ is decreasing. Then define $r_k := \dfrac{\|a_k\|}{2}$ and $R(\varepsilon_k) := \dfrac{1}{\varepsilon_k}$. It follows from the constructions above that

$$r_k R(r_k) \le r'_k \varepsilon_k \frac{1}{\varepsilon_k} = r'_k, \quad k \in I\!\!N.$$

We clearly see that the sequence $\{R(r_k)\}$ is increasing as $r_k \downarrow 0$. Extending $R(\cdot)$ piecewise linearly to $\mathbb{R}_+$ brings us to the framework of Definition 4.8 and thus completes the proof of the proposition. $\square$

Finally in this section, we show the rated extremality conditions of Definition 4.3 holds for R-perturbed extremal points of infinite set systems from Definition 4.8.

Theorem 4.10 (Rated Extremal Principle for Perturbed Systems)**.** *Let $\bar{x}$ be an R-perturbed extremal point of a closed set system $\{\O_i\}_{i \in T}$ in an Asplund space X. Then the rated extremal principle holds for this system at $\bar{x}$.*

Proof. Fix $\varepsilon > 0$ and find I, $\{x_i\}_{i \in I}$, and $\{a_i\}_{i \in I}$ from Definition 4.8 such that

$$\bigcap_{i \in I} \big(\O_i - x_i - a_i\big) \cap (rR) I\!\!B = \emptyset.$$

For convenience denote $I := \{1, \ldots, N\}$ and define

$$\O := \Big\{(u_1, \ldots, u_N) \in X^N \,\Big|\, u_i \in \O_i \cap (x_i + rR I\!\!B),\ i \in I\Big\}.$$

For any $z = (u_1, \ldots, u_N) \in \emptyset$ consider the function

$$\varphi(z) := \Big(\sum_{i=2}^{N} \|(u_1 - x_1 - a_1) - (u_i - x_i - a_i)\|^2 \Big)^{\frac{1}{2}} > 0.$$

Furthermore, for $\bar{z} = (x_1, \ldots, x_N)$ we have the estimates

$$\varphi(\bar{z}) = \Big(\sum_{i=2}^{N} \|a_1 - a_i\|^2 \Big)^{\frac{1}{2}} < 2r\sqrt{N} \le \inf_{z \in \emptyset} \varphi(z) + 2rN^{\frac{1}{2}}.$$

The rest of the proof follows the arguments in the proof of Theorem 4.6. $\square$

5 Calculus Rules for Rated Normals to Infinite Intersections

In the concluding section of the paper we apply the rated extremal principle of Section 4 to deriving some calculus rules for general normals to infinite set intersections, which are closely related to necessary optimality conditions in problems of semi-infinite and infinite programming. Unless otherwise stated, the spaces below are Asplund and the sets under consideration are closed around reference points. As in Section 4, we often drop the subscript "r" for simplicity in the notation of rate functions $R_r = R(r)$ if no confusion arises. In addition, we always assume that rate functions are continuous.

We start with the following definition of *rated normals* to set intersections.

Definition 5.1 (Rated normals to set intersection). *Let $\emptyset := \bigcap_{i \in T} \emptyset_i$, and let $\bar{x} \in \emptyset$. We say that a dual element $x^* \in X^*$ is an R-NORMAL to the set intersection $\emptyset$ if for any $r \downarrow 0$ there is $I = I(r) \subset T$ of cardinality $|I|^{3/2} = o(R_r)$ such that*

$$\langle x^*, x - \bar{x} \rangle - r\|x - \bar{x}\| < r \quad \text{for all} \quad x \in \bigcap_{i \in I} \emptyset_i \cap B(\bar{x}, rR_r). \tag{5.1}$$

The next proposition reveals relationships between Fréchet and R-normals to set intersections.

Proposition 5.2 (Rated normals versus Fréchet normals to set intersections). *Let $\bar{x} \in \emptyset = \bigcap_{i \in I} \emptyset_i$. Then any R-normal to $\emptyset$ at $\bar{x}$ is a Fréchet normal to $\emptyset$ at $\bar{x}$. The converse holds if I is finite.*

Proof. Assume x^* is an R-normal to $\emptyset$ at $\bar{x}$ with some rate function $R(r)$ while x^* is not a Fréchet normal to $\emptyset$ at this point. Hence there are $\delta > 0$ and a sequence $x_k \xrightarrow{\emptyset} \bar{x}$ such that $\delta\|x_k - \bar{x}\| < \langle x^*, x_k - \bar{x} \rangle$ for all $k \in I\!N$. Hence $x_k \ne \bar{x}$ and

$$\delta\|x_k - \bar{x}\| < \langle x^*, x_k - \bar{x} \rangle < r\|x_k - \bar{x}\| + r$$

whenever $\|x_k - \bar{x}\| \le rR$. Now suppose that $rR = M > 0$ for some M and then fix a number $k \in I\!N$ such that $\|x_k - \bar{x}\| \le rR$. Letting $r \downarrow 0$, we arrive at the contradiction $\delta\|x_k - \bar{x}\| \le 0$.

Consider next the remaining case when $rR \to 0$ as $r \downarrow 0$ and find $r_k > 0$ sufficiently small so that $\|x_k - \bar{x}\| = r_k R(r_k)$ due to the continuity of R and the convergence $rR \xrightarrow{r \downarrow 0} 0$. It follows that

$$\delta r_k R(r_k) < r_k^2 R(r_k) + r_k \quad \text{and hence} \quad \delta < r_k + \frac{1}{R(r_k)}, \quad k \in I\!N,$$

which gives a contradiction as $k \to \infty$. Thus x^* is a Fréchet normal to $\emptyset$ at $\bar{x}$.

Conversely, assume that the index set I is finite, i.e., $I = \{1, \ldots, N\}$, and that x^* is a Fréchet normal. Then for any $r > 0$ we have by (2.1) that

$$\langle x^*, x - \bar{x} \rangle - r\|x - \bar{x}\| \le 0 \quad \text{for all} \quad x \in \bigcap_{i=1}^{N} \emptyset_i \cap U,$$

where U is a neighborhood of $\bar{x}$. This clearly implies (5.1) with any rate function R, which ensures that x^* is an R-normal to $\emptyset$ at $\bar{x}$ and thus completes the proof of the proposition. $\square$

The next example concerns infinite systems of convex sets in $\mathbb{R}^2$. It illustrates the way of computing R-normals to infinite intersections and shows that R-normals in this case reduce to usual ones.

Example 5.3 (Rated normals for infinite systems)**.** Let $m \geq 4$ be a fixed integer. Consider an infinite system of convex sets $\{\emptyset_k\}_{k \in I\!N}$ in $\mathbb{R}^2$ defined as the epigraphs of the convex and smooth functions

$$g_k(x) := \begin{cases} k^m x^2 & \text{for } \ x \geq 0, \\ 0 & \text{for } \ x < 0, \end{cases} \qquad k = 1, 2, \ldots.$$

Let $\bar{x} := (0,0)$, $\emptyset := \bigcap_{k=1}^{\infty} \emptyset_k$, and let $R = R(r) = r^{\alpha - 1}$ for some $\alpha \in (0, \frac{2}{11})$. We obviously get $\emptyset = \mathbb{R}_- \times \mathbb{R}_+$ and $N(\bar{x}; \emptyset) = \mathbb{R}_+ \times \mathbb{R}_-$. Let us verify that $x^* = (1,0)$ is an R-normal to $\emptyset$ at $\bar{x}$, which implies the whole normal cone $N(\bar{x}; \emptyset)$ consists of R-normals.

To proceed, fix any $r > 0$ sufficiently small and denote by k_0 the *smallest* integer such that

$$\max\Big\{\frac{1}{4r^2}, \frac{1}{4r^{2+\alpha}}\Big\} = \frac{1}{4r^{2+\alpha}} \leq k_0^m.$$

Now consider $I := \{1, \ldots, k_0\}$ and check that

$$k_0 \leq \Big(\frac{1}{4r^{2+\alpha}}\Big)^{1/m} + 1 < \frac{1}{r^{\frac{2+\alpha}{m}}}.$$

Since $1 - \frac{3}{2m}(2+\alpha) - \alpha \geq 1 - \frac{3}{8}(2+\alpha) - \alpha \geq \frac{1}{4} - \frac{11}{8}\alpha > 0$, it follows that

$$\frac{|I|^{3/2}}{R} < \frac{r^{1-\alpha}}{r^{\frac{3(2+\alpha)}{2m}}} = r^{1 - \frac{3}{2m}(2+\alpha) - \alpha} \to 0 \quad \text{when} \quad r \downarrow 0.$$

Defining further $\emptyset_0 := \bigcap_{k=1}^{k_0} \emptyset_k$, it remains to show that

$$\langle x^*, x \rangle - r\|x\| < r \ \ \text{for all} \ \ x \in \emptyset_0 \cap B(0; rR). \tag{5.2}$$

To verify (5.2), take $x := (t,s)$ and consider only the case when $t > 0$, since the other case of $t \leq 0$ is obvious. For $t > 0$ we have $s \geq k_0^m t^2$ and

$$\langle x^*, x \rangle - r\|x\| = t - r\sqrt{t^2 + s^2} \leq t\Big(1 - r\sqrt{1 + k_0^{2m} t^2}\Big) < t\big(1 - r k_0^m t\big) = -r k_0^m t^2 + t =: f(t). \tag{5.3}$$

It follows from $\|x\| \leq rR = r^\alpha$ that

$$r^\alpha \geq \sqrt{t^2 + s^2} \geq t\sqrt{1 + k_0^{2m} t^2} > k_0^m t^2$$

and hence $t < \big(\frac{r^\alpha}{K^m}\big)^{1/2}$. The latter implies that for all $x = (t,s) \in \emptyset_0 \cap B(0; rR)$ with $t > 0$ we have

$$\langle x^*, x \rangle - r\|x\| < f(t) \leq \sup_{[0,a]} f(t) \quad \text{with} \quad a := \Big(\frac{r^\alpha}{k_0^m}\Big)^{1/2} \geq \frac{1}{2rk_0^m}.$$

Observe finally that the function $f(t)$ in (5.3) attains its maximum on [0,a] at the point $t = \frac{1}{2rk_0^m}$ and that

$$\sup_{[0,a]} f(t) = -rk_0 \frac{1}{4r^2 k_0^{2m}} + \frac{1}{2rk_0^m} = \frac{1}{4rk_0^m} \leq r.$$

Combining all the above, we arrive at (5.2) and thus achieve our goals in this example.

The next example related to the previous one involves the notion of equicontinuity for systems of mappings. Given $f_i \colon X \to Y$, $i \in T$, we say that the system $\{f_i\}_{i \in T}$ is *equicontinuous* at $\bar{x}$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that $\|f_i(x) - f_i(\bar{x})\| < \varepsilon$ for all $x \in B(\bar{x}, \delta)$ and $i \in T$. This notion has been recently exploited in [15] in the framework of variational analysis; see Remark 5.14.

Example 5.4 (Non-equicontinuity of gradient and normal systems)**.** Given an integer $m \ge 4$, define an infinite systems of functions $\varphi_k\colon \mathbb{R}^2 \to \mathbb{R}$ for $k \in I\!N$ by

$$\varphi_k(x_1, x_2) := \begin{cases} k^m x_1^2 - x_2 & \text{for } \; x_1 > 0, \\ -x_2 & \text{for } \; x_1 \le 0. \end{cases} \tag{5.4}$$

It is easy to check that the system of gradients $\{\nabla \varphi_k\}_{k \in I\!N}$ is not equicontinuous at $\bar{x} = (0, 0)$.

Furthermore, observe that the sets $\O_k$ in Example 5.3 can be defined by

$$\O_k := \big\{x \in \mathbb{R}^2 \big|\ \varphi_k(x) \le 0\big\}, \quad k \in I\!N. \tag{5.5}$$

Given any boundary point (x_1, x_2) of the set $\O_k$, we compute the unit normal vector to $\O_k$ at (x_1, x_2) by

$$\xi_k(x_1, x_2) = \begin{cases} \dfrac{1}{\sqrt{4k^{2m}x_1^2 + 1}}(2k^m x_1, -1) & \text{for } \; x_1 > 0, \\ (0, -1) & \text{for } \; x_1 \le 0. \end{cases}$$

and then check the relationships for $x_1 > 0$:

$$\|\xi_k(x_1, x_2) - \xi_k(0, 0)\|^2 = \frac{8k^{2m}x_1^2 - 2\sqrt{4k^{2m}x_1^2 + 1}}{4k^{2m}x_1^2 + 1} \to 2 \;\text{ as }\; k \to \infty.$$

The latter means that the system of $\{\xi_k\}_{k \in I\!N}$ is not equicontinuous at $\bar{x} = (0, 0)$.

The next major result of this paper establishes a certain "fuzzy" intersection rule for rated normals to infinite set intersections. Its proof is based on the rated extremal principle for infinite set systems obtained above in Theorem 4.6. Parts of this proof are similar to deriving a fuzzy sum rule for Fréchet normals to intersections of two sets in Asplund spaces given in [12] and in [8, Lemma 3.1] on the base of the approximate extremal principle for such set systems.

Theorem 5.5 (Fuzzy intersection rule for R-normals)**.** *Let $\bar{x} \in \O := \bigcap_{i \in T} \O_i$, and let $x^* \in X^*$ be an R-normal to $\O$ at $\bar{x}$. Then for any $\varepsilon > 0$ there exist an index subset I, Fréchet normals $x_i^* \in \widehat{N}(x_i; \O_i)$ with $\|x_i - \bar{x}\| < \varepsilon$ for $i \in I$, and a number $\lambda \ge 0$ such that*

$$\lambda x^* \in \sum_{i \in I} x_i^* + \varepsilon I\!B^* \;\; \text{and} \;\; \lambda^2 + \lambda^2\|x^*\|^2 + \sum_{i \in I}\|x_i^*\|^2 = 1. \tag{5.6}$$

Proof. Without loss of generality, assume that $\bar{x} = 0$. Pick any $x^* \in \widehat{N}(0; \O)$ and by Definition 5.1 for any $r > 0$ sufficiently small find an index subset $|I|^{3/2} = o(R)$ such that

$$\langle x^*, x\rangle - r\|x\| < r \;\text{ whenever }\; x \in \bigcap_{i \in I} \O_i \cap (rR)I\!B. \tag{5.7}$$

Then we form the following closed subsets of the Asplund space $X \times \mathbb{R}$:

$$\begin{aligned} &O_1 := \Big\{(x, \alpha) \in X \times \mathbb{R} \Big|\ x \in \O_1,\ \alpha \le \langle x^*, x\rangle - r\|x\|\Big\}, \\ &O_i := \O_i \times \mathbb{R}_+ \;\text{ for }\; i \in I \setminus \{1\}, \end{aligned} \tag{5.8}$$

where $I = \{1, \ldots, N\}$ with "1" denoting the first element of I for simplicity. This leads us to

$$\Big(O_1 - (0, r)\Big) \cap \bigcap_{i \in I \setminus \{1\}} O_i \cap (rR_r)I\!B = \emptyset. \tag{5.9}$$

Indeed, if on the contrary (5.9) does not hold, we get (x, α) from the above intersection satisfying $\alpha \ge 0$, $x \in \bigcap_{i \in I} \O_i \cap (\varepsilon R_\varepsilon)I\!B$, and

$$r \le \alpha + r \le \langle x^*, x\rangle - r\|x\|,$$

where the latter is due to $(x,\alpha+r)\in O_1$. This clearly contradicts (5.7) and so justifies (5.9). Thus we have that $(0,0)\in X\times\mathbb{R}$ is a rated extremal point of the set system $\{O_1,O_2\}$ from (5.8) in the sense of Definition 4.1. Applying to this system the rated extremal principle from Theorem 4.6 with taking into account Remark 4.7 to find elements (w_i,α_i) and $(x_i^*\lambda_i)$ for $i=1,\ldots,N$ satisfying the relationships

$$\left\{\begin{array}{l}(x_i^*,\lambda_i)\in\widehat{N}\big((w_i,\alpha_i);O_i\big),\ \|(w_i,\alpha_i)\|\le 2rR^{\frac{1}{2}}N^{\frac{3}{4}},\quad i\in I,\\ \Big\|(x_1^*,\lambda_1)+\ldots+(x_N^*,\lambda_N)\Big\|\le\dfrac{4N^{\frac{3}{4}}}{R^{\frac{1}{2}}}=:\eta\downarrow 0\ \text{ as }\ r\downarrow 0,\\ \|(x_1^*,\lambda_1)\|^2+\ldots+\|(x_N^*,\lambda_N)\|^2=1.\end{array}\right. \tag{5.10}$$

By the structure of O_i as $i=1,\ldots,N$ we have from the first line of (5.10) that $x_i^*\in\widehat{N}(w_i;\O_i)$, that $\lambda_i\le 0$ for $i=2,\ldots,N$, and that

$$\limsup_{(x,\alpha)\stackrel{O_1}{\to}(w_1,\alpha_1)}\frac{\langle x_1^*,x-w_1\rangle+\lambda_1(\alpha-\alpha_1)}{\|x-w_1\|+|\alpha-\alpha_1|}\le 0 \tag{5.11}$$

by the definition of Fréchet normals. It also follows from the structure of O_1 that $\lambda_1\ge 0$ and

$$\alpha_1\le\langle x^*,w_1\rangle-r\|w_1\|. \tag{5.12}$$

This allows us to split the situation into the follows two cases.

Case 1: $\lambda_1=0$. If inequality (5.12) is strict in this case, there is a neighborhood W of w_1 such that

$$\alpha_1\le\langle x^*,x\rangle-r\|x\|\ \text{ for all }\ x\in\O_1\cap W.$$

This implies that $(x,\alpha_1)\in O_1$ whenever $x\in\O_1\cap W$. Substituting (x,α_1) into (5.11) gives us

$$\limsup_{x\stackrel{\O_1}{\to}w_1}\frac{\langle x_1^*,x-w_1\rangle}{\|x-w_1\|}\le 0,\ \text{ i.e., }\ x_1^*\in\widehat{N}(w_1;\O_1).$$

If (5.12) holds as equality, we denote $\alpha:=\langle x^*,x\rangle-r\|x\|$ and get

$$|\alpha-\alpha_1|=\Big|\langle x^*,x-w_1\rangle+r(\|w_1\|-\|x\|)\Big|\le\Big(\|x^*\|+r\Big)\|x-w_1\|,$$

which implies by (5.11) that

$$\limsup_{(x,\alpha)\stackrel{O_1}{\to}(w_1,\alpha_1)}\frac{\langle x_1^*,x-w_1\rangle}{\|x-w_1\|+|\alpha-\alpha_1|}\le 0.$$

Thus it follows for any $\varepsilon'>0$ sufficiently small and the number α chosen above that

$$\langle x_1^*,x-w_1\rangle\le\varepsilon'\Big(\|x-w_1\|+|\alpha-\alpha_1|\Big)\le\varepsilon'\Big(1+\|x^*\|+r\Big)\|x-w_1\|$$

for all $x\in\O_1$ sufficiently closed to w_1. This ensures that

$$\limsup_{x\stackrel{\Lambda_1}{\to}w_1}\frac{\langle x_1^*,x-w_1\rangle}{\|x-w_1\|}\le 0,\ \text{ i.e., }\ x_1^*\in\widehat{N}(w_1;\O_1)$$

when (5.12) holds as equality as well as the strict inequality. Since $\lambda_1=0$ in Case 1 under consideration and since $\lambda_i\le 0$ for all $i\ge 2$, it follows that

$$\lambda_2^2+\ldots+\lambda_N^2\le(\lambda_2+\ldots+\lambda_N)^2\le\eta^2.$$

This leads us to the estimates

$$\|x_1^*\|^2+\ldots+\|x_N^*\|^2\ge 1-(\lambda_2^2+\ldots+\lambda_N^2)\ge\frac{1}{2},$$

and thus we get from (5.10) all the conclusion of the theorem with $\lambda=0$ in (5.6) in this case.

Case 2: $\lambda_1 > 0$. If inequality (5.12) is strict in this case, put $x := w_1$ and get from (5.11) that

$$\limsup_{\alpha \to \alpha_1} \frac{\lambda_1(\alpha - \alpha_1)}{|\alpha - \alpha_1|} \le 0,$$

which yields $\lambda_1 = 0$, a contradiction. It remains therefore to consider the case when (5.12) holds as equality. Take then a pair $(x, \alpha) \in O_1$ with

$$x \in \varnothing_1 \setminus \{w_1\} \quad \text{and} \quad \alpha = \langle x^*, x \rangle - r\|x\|$$

and hence get from (5.12) that

$$\alpha - \alpha_1 = \langle x^*, x - w_1 \rangle + r(\|w_1\| - \|x\|),$$

which implies the relationships

$$\langle x_1^*, x - w_1 \rangle + \lambda_1(\alpha - \alpha_1) = \langle x_1^* + \lambda_1 x^*, x - w_1 \rangle + \lambda_1 r(\|w_1\| - \|x\|),$$

$$|\alpha - \alpha_1| \le (\|x^*\| + r)\|x - w_1\|.$$

On the other hand, it follows from (5.11) that for any $\varepsilon' > 0$ sufficiently small there exists a neighborhood V of w_1 such that

$$\langle x_1^*, x - w_1 \rangle + \lambda_1(\alpha - \alpha_1) \le \lambda_1 \varepsilon' r\Big(\|x - w_1\| + |\alpha - \alpha_1|\Big),$$

whenever $x \in \varnothing_1 \cap V$ and that

$$\begin{aligned}\langle x_1^* + \lambda_1 x^*, x - w_1 \rangle + \lambda_1 r(\|w_1\| - \|x\|) &\le \lambda_1 \varepsilon' r(\|x - w_1\| + |\alpha - \alpha_1|)\\ &\le \lambda_1 \varepsilon' r\Big[\|x - w_1\| + (\|x^*\| + r)\|x - w_1\|\Big]\\ &= \lambda_1 \varepsilon' r\big(1 + \|x^*\| + r\big)\|x - w_1\|.\end{aligned}$$

Let us now choose $\varepsilon' > 0$ sufficiently small so that

$$\langle x_1^* + \lambda_1 x^*, x - w_1 \rangle + \lambda_1 r(\|w_1\| - \|x\|) \le \lambda_1 r\|x - w_1\|.$$

and for all $x \in \varnothing_1 \cap V$ get the estimate

$$\langle x_1^* + \lambda_1 x^*, x - w_1 \rangle \le \lambda_1 r\|x - w_1\| + \lambda_1 r(\|x\| - \|w_1\|) \le 2\lambda_1 r\|x - w_1\|.$$

It follows definition (2.1) of ε-normals that

$$x_1^* + \lambda_1 x^* \in \widehat{N}_{2\lambda_1 r}(w_1; \varnothing_1),$$

where $\lambda_1 \le 1$ by the third line of (5.10). Using the representation of ε-normals in Asplund spaces from [8, (2.51)], we find $v \in \varnothing_1 \cap (w_1 + 2\lambda_1 r)I\!B)$ such that

$$x_1^* + \lambda_1 x^* \in \widehat{N}(v; \varnothing_1) + 2\lambda_1 r I\!B^*.$$

Hence $\|v\| \le \|v - w_1\| + \|w_1\| \le 2\lambda_1 r + 2rR^{\frac{1}{2}}N^{\frac{3}{4}} \le 3rR^{\frac{1}{2}}N^{\frac{3}{4}}$ and there is $\widetilde{x}_1^* \in \widehat{N}(v; \varnothing_1)$ with

$$\lambda_1 x^* \in \widetilde{x}_1^* - x_1^* + 2\lambda_1 r I\!B^*.$$

Taking into account that $x_1^* + \ldots + x_N^* \in \eta I\!B^*$, we get

$$\lambda_1 x^* \in \widetilde{x}_1^* + x_2^* + \ldots + x_N^* + (2\lambda_1 r + \eta)I\!B^*.$$

On the other hand, it follows from $-x_1^* = \lambda_1 x^* - \widetilde{x}_1^* - u^*$ with some $\|u^*\| \le 2\lambda_1 r \le 2r$ that

$$\|x_1^*\|^2 \le \big(\lambda_1\|x^*\| + \|\widetilde{x}_1^*\| + 2r\big)^2 \le 2\lambda_1^2\|x^*\|^2 + 2\|\widetilde{x}_1^*\|^2 + \frac{1}{4}.$$

Moreover, since $|\lambda_1+\lambda_2+\ldots+\lambda_N|\le\eta\downarrow 0$ as $r\downarrow 0$ by the second line of (5.10) and since $\lambda_1\ge 0$ while $\lambda_i\le 0$ for $i=2,\ldots,N$, we have

$$\eta^2>\lambda_1^2+(\lambda_2+\ldots+\lambda_N)^2+2\lambda_1(\lambda_2+\ldots+\lambda_N)>\lambda_1^2+(\lambda_2+\ldots+\lambda_N)^2+2\lambda_1(-\lambda_1-\eta)$$

It also follows from (5.10) and $0<\lambda_1<1$ that

$$\lambda_1^2\ge(\lambda_2+\ldots\lambda_N)^2-\eta^2-2\eta\lambda_1\ge\lambda_2^2+\ldots+\lambda_N^2-\frac{1}{4},$$

which leads us to the subsequent estimates

$$\lambda_1^2+\ldots+\lambda_N^2\le 2\lambda_1^2+\frac{1}{4}\ \text{ and}$$

$$\begin{aligned}1&\le\Big(\lambda_1^2+\ldots+\lambda_N^2\Big)+\Big(\|x_1^*\|^2+\ldots+\|x_N^*\|^2\Big)\\&\le 2\lambda_1^2+2\lambda_1^2\|x^*\|^2+2\|\tilde{x}_1^*\|^2+\Big(\|x_2^*\|^2+\ldots+\|x_N^*\|^2\Big)+\frac{1}{2}.\end{aligned}$$

This finally ensures that

$$\frac{1}{4}\le\lambda_1^2+\lambda_1^2\|x^*\|^2+\|\widetilde{x}_1^*\|^2+\|x_2^*\|^2+\ldots+\|x_N^*\|^2$$

and brings us to all the conclusions of the theorem with $\lambda:=\lambda_1$ in (5.6). $\square$

Remark 5.6 (Quantitative estimates in the intersection rule)**.** It can be observed directly from the proof of Theorem 5.5 that we get in fact the following quantitative estimates in intersection rule obtained for infinite set systems when $r>-$ is sufficiently small: $|I|^{3/2}=o(R)$,

$$\|x_i-\bar{x}\|<3rR^{\frac{1}{2}}|I|^{\frac{3}{4}},\ \text{ and }\ \lambda x^*\in\sum_{i\in I}x_i^*+\Big(2r+4\frac{|I|^{\frac{3}{4}}}{R^{\frac{1}{2}}}\Big)I\!B^*.$$

In particular, for $R=O\big(\frac{1}{r}\big)$, there is $C>0$ such that all the conclusions hold with $|I|^{3/2}=N^{3/2}=o\big(\frac{1}{r}\big)$,

$$\|x_i-\bar{x}\|<C\sqrt{rN^{\frac{3}{2}}},\ \text{ and }\ \lambda x^*\in\sum_{i\in I}x_i^*+C\sqrt{rN^{\frac{3}{2}}}I\!B^*.$$

Remark 5.7 (Perturbed rated normals to infinite intersections)**.** Inspired by our consideration of perturbed extremal systems in Section 4, we define a perturbed version of R-normals to infinite set intersections as follows: $x^*\in X^*$ is a *perturbed R-normal* to the intersection $\emptyset:=\bigcap_{i\in T}\emptyset_i$ at $\bar{x}\in\emptyset$ if for any $\varepsilon>0$ there exist a number $r>0$, an index subset I with cardinality $|I|^{3/2}=o(R_r)$, and points $x_i\in\emptyset_i\cap B(\bar{x},\varepsilon)$ as $i\in I$ such that $r|I|<\varepsilon$ and

$$\langle x^*,x\rangle-r\|x\|<r\ \text{ whenever }\ x\in\bigcap_{i\in I}\big(\emptyset_i-x_i\big)\cap(rR_r)I\!B.$$

Then the corresponding version of the intersection rule from Theorem 5.5 can be derived for perturbed rated normals to infinite intersections by a similar way with replacing in the proof the rated extremal principle from Theorem 4.6 by its perturbed version from Theorem 4.10.

We proceed with deriving calculus rules for the so-called *limiting R-normals* (defined below) to infinite intersections of sets. First we propose a new qualification conditions for infinite systems.

Definition 5.8 (Approximate qualification condition)**.** *We say that a system of sets $\{\emptyset_i\}_{i\in T}\subset X$ satisfies the* APPROXIMATE QUALIFICATION CONDITION (AQC) *at $\bar{x}\in\bigcap_{i\in T}\emptyset_i$ if for any $\varepsilon\downarrow 0$, any finite index subset $I_\varepsilon\subset T$, and any Fréchet normals $x_{i\varepsilon}^*\in\widehat{N}(x_{i\varepsilon};\emptyset_i)\cap I\!B^*$ with $\|x_{i\varepsilon}-\bar{x}\|\le\varepsilon$ as $i\in I_\varepsilon$ the following implication holds:*

$$\Big\|\sum_{i\in I_\varepsilon}x_{i\varepsilon}^*\Big\|\stackrel{\varepsilon\downarrow 0}{\longrightarrow}0\Longrightarrow\sum_{i\in I_\varepsilon}\|x_{i\varepsilon}^*\|^2\stackrel{\varepsilon\downarrow 0}{\longrightarrow}0.\tag{5.13}$$

The next proposition presents verifiable conditions ensuring the validity of AQC for finite systems of sets under the SNC property (3.13) discussed at the end of Section 3; see [8] for more details.

Proposition 5.9 (AQC for finite set systems under SNC assumptions)**.** *Let $\{\Omega_1,\ldots,\Omega_m\}$ be a finite set system satisfying the limiting qualification condition at $\bar{x}\in\bigcap_{i=1}^m\Omega_i$: for any sequences $x_{ik}\stackrel{\Omega_i}{\to}\bar{x}$ and $x_{ik}^*\stackrel{w^*}{\to}x_i^*$ with $x_{ik}^*\in\widehat{N}(x_{ik};\Omega_i)$ as $k\to\infty$ and $i=1,\ldots,m$ we have*

$$\|x_{1k}^*+\ldots+x_{mk}^*\|\to 0\Longrightarrow x_1^*=\ldots=x_m^*=0,$$

which is automatic under the normal qualification condition via the basic normal cone (2.2)*:*

$$\big[x_1^*+\ldots+x_m^*=0\ \text{ and }\ x_i^*\in N(\bar{x};\Omega_i),\ i=1,\ldots,m\big]\Longrightarrow x_i^*=0\ \text{ for all }\ i=1,\ldots,m.$$

Assume in addition that all but one of Ω_i are SNC at $\bar{x}$. Then the AQC is satisfied for $\{\Omega_1,\ldots,\Omega_m\}$ at $\bar{x}$.

Proof. Pick $\varepsilon_k\downarrow 0$, $x_{ik}^*\in\widehat{N}(x_{ik};\Omega_i)\cap I\!\!B^*$, $\|x_{ik}-\bar{x}\|\le\varepsilon_k$ as $i=1,\ldots,m$ and assume that

$$\|x_{1k}^*+\ldots+x_{mk}^*\|\to 0\ \text{ as }\ k\to\infty. \tag{5.14}$$

Taking into account that the sequences $\{x_{ik}^*\}\subset X^*$ are bounded when X is Asplund, we extract from them weak* convergent subsequences and suppose with no relabeling that $x_{ik}^*\stackrel{w^*}{\to}x_i^*$ as $k\to\infty$ for all $i=1,\ldots,m$. It follows from the imposed limiting qualification condition for $\{\Omega_1,\ldots,\Omega_m\}$ at $\bar{x}$ that $x_1^*=\ldots=x_m^*=0$. Since all but one (say for $i=1$) of the sets Ω_i are SNC at $\bar{x}$, we have that $\|x_{ik}^*\|\to 0$ as $k\to\infty$ for $i=2,\ldots,m$. Then (5.14) implies that $\|x_{1k}^*\|\to 0$ as well, which verifies implication (5.13) and thus completes the proof of the proposition. $\square$

The following example illustrates the validity of the AQC for infinite systems of sets.

Example 5.10 (AQC for infinite systems)**.** We verify that the AQC holds in the framework of Example 5.4 at the origin $\bar{x}=(0,0)\in\mathbb{R}^2$. Recall that for each $k\in I\!\!N$ the normal cone to a convex set Ω_k from (5.5) at a boundary point $x=(x_1,x_2)$ is computed by

$$N(x;\Omega_k)=\mathbb{R}_+\xi_k(x)\ \text{ with }\ \xi_k(x)=\xi_k(x_1,x_2)=\begin{cases}(2k^m x_1,-1) & \text{for } x_1>0,\\ (0,-1) & \text{for } x_1\le 0.\end{cases}$$

If according to the left-hand side of (5.13) we have

$$\Big\|\sum_{k\in I_\varepsilon}\lambda_{\varepsilon k}\xi_k(x_{\varepsilon k})\Big\|\to 0\ \text{ as }\ \varepsilon\downarrow 0,$$

then it follows from the above representation of ξ_k that its component goes to zero as $k\to\infty$. Thus

$$\sum_{k\in I_\varepsilon}\|\lambda_{\varepsilon k}\xi_k(x_{\varepsilon k})\|^2\to 0\ \text{ as }\ \varepsilon\downarrow 0,$$

which verifies the AQC property of the system $\{\Omega_k\}_{k\in I\!\!N}$ at $\bar{x}$.

Now we are ready to define limiting R-normals and derive infinite intersection rules for them. In the definition below R_k stands for a rate function for each x_k^*; these functions may be different from each other.

Definition 5.11 (Limiting R-normals to infinite set intersections)**.** *Consider an arbitrary set system $\{\Omega_i\}_{i\in T}\subset X$, and let $\Omega:=\bigcap_{i\in T}\Omega_i$ with $\bar{x}\in\Omega$. We say that a dual element x^* is a* LIMITING R-NORMAL *to Ω at $\bar{x}$ if there exist sequences $\{(x_k,x_k^*)\}_{k\in I\!\!N}\subset X\times X^*$ such that $x_k\stackrel{\Omega}{\to}\bar{x}$, $x_k^*\stackrel{w^*}{\longrightarrow}x^*$ as $k\to\infty$ and that each element x_k^* is an R_k-normal to Ω at x_k,*

It is clear from the definition and Proposition 5.2 that any limiting R-normal is a basic/limiting normal to $\mathcal{O}$ at $\bar{x}$. Conversely, if T is a finite index set and X is an Asplund space, then we the reverse implication holds, i.e., any limiting/basic normal is a limiting R-normal.

The next theorem provides a representation of limiting R-normals to infinite set intersections via Fréchet normals to each set under consideration. In particular, it implies a useful calculus rule for the basic normal cone (2.2) to infinite intersections.

Theorem 5.12 (Representation of limiting R-normals to infinite intersections). *Let $\mathcal{O} := \bigcap_{i\in T}\mathcal{O}_i$ with $\bar{x}\in\mathcal{O}$ for the system $\{\mathcal{O}_i\}_{i\in T}\subset X$ satisfying the AQC property from Definition* 5.8 *at $\bar{x}$. Then for any given limiting R-normal to $\mathcal{O}$ at $\bar{x}$ and any $\varepsilon>0$ we have the inclusion*

$$x^*\in \operatorname{cl}^*\Big\{\sum_{i\in I}x_i^*+\varepsilon I\!B^*\;\Big|\;x_i^*\in\widehat{N}(x_i;\mathcal{O}_i),\;\|x_i-\bar{x}\|<\varepsilon,\;I\subset T\Big\},$$

where $I\subset T$ is a finite index subset. In particular, if all the limiting/basic normals to $\mathcal{O}$ at $\bar{x}$ are limiting R-normals in this setting, then

$$N(\bar{x};\mathcal{O})\subset\bigcap_{\varepsilon>0}\operatorname{cl}^*\Big\{\sum_{i\in I}x_i^*+\varepsilon I\!B^*\;\Big|\;x_i^*\in\widehat{N}(x_i;\mathcal{O}_i),\;\|x_i-\bar{x}\|<\varepsilon,\;I\subset T\Big\}.\tag{5.15}$$

Proof. Take a sequence $\{x_k^*\}$ of R-normals to $\mathcal{O}$ at x_k with $x_k\to\bar{x}$ and $x_k^*\stackrel{w^*}{\to}x^*$ as $k\to\infty$. The latter convergence ensures by the Uniform Boundedness Principle that the set $\{\|x_k^*\|\}_{k\in I\!N}$ is bounded in X^*. Picking $\varepsilon>0$ sufficiently small, we find $x_k\in\mathcal{O}$ with $\|x_k-\bar{x}\|<\varepsilon$. Applying Theorem 5.5 to x_k^* for each $k\in I\!N$ gives us sequences $x_{ik}^*\in\widehat{N}(x_{ik};\mathcal{O}_i)$ with $\|x_{ik}-x_k\|<\varepsilon$ for $i\in I_k\subset T$ and $\lambda_k\ge 0$ satisfying

$$\lambda_k x_k^*\in\sum_{i\in I_k}x_{ik}^*+\varepsilon I\!B^*\;\text{ and }\;\lambda_k^2+\lambda_k^2\|x_k^*\|^2+\sum_{i\in I_k}\|x_{ik}^*\|^2=1,\quad k\in I\!N.\tag{5.16}$$

Let us show that the sequence $\{\lambda_k\}$ is bounded away from 0. Assuming on the contrary $\lambda_k\downarrow 0$ as $k\to\infty$, we have

$$\Big\|\sum_{i\in I_k}x_{ik}^*\Big\|\longrightarrow 0\text{ as }\;k\to\infty$$

from the inclusion in (5.16). Then the imposed AQC leads us to

$$\sum_{i\in I_k}\|x_{ik}^*\|^2\to 0\;\text{ as }\;k\to\infty,$$

which contradicts the equality in (5.16) and thus shows that there is constant $C>0$ with $\lambda_k>C$ for all $k\in I\!N$ sufficiently large. Rescaling finally the inclusion in (5.16), we get

$$x_k^*\in\sum_{i\in I}\frac{x_{ik}^*}{\lambda_k}+\frac{\varepsilon}{C}I\!B^*,\quad k\in I\!N,$$

which ensures that $x_k^*\xrightarrow{w^*}x^*$ as $k\to\infty$ and thus justifies the first conclusion of the theorem. The second ones on basic normals follows immediately. $\square$

The next corollary provides more explicit results for the case of infinite systems of cones, with the replacement of Fréchet normals in Theorem 5.12 by basic normals at the origin.

Corollary 5.13 (Limiting R-normals to intersection of cones). *Let $\{\Lambda_i\}_{i\in T}$ be a system of cones in X, and let $\Lambda := \bigcap_{i\in T}\Lambda_i$. Suppose that $x^*\in X^*$ is a limiting R-normal to Λ at the origin and that the AQC property from Definition* 5.8 *holds at $\bar{x}=0$. Then for any $\varepsilon>0$ we have the representation*

$$x^*\in\operatorname{cl}^*\Big\{\sum_{i\in I}x_i^*+\varepsilon I\!B^*\;\Big|\;x_i^*\in N(0;\Lambda_i),\;I\subset T\Big\}$$

via finite index subsets $I \subset T$. If furthermore all the limiting/basic normals to Λ at the original are limiting R-normals in this setting, then

$$N(0;\Lambda) \subset \bigcap_{\varepsilon>0} \mathrm{cl}^*\Big\{ \sum_{i\in I} x_i^* + \varepsilon I\!B^* \;\Big|\; x_i^* \in N(0;\Lambda_i),\ I \subset T \Big\}.$$

Proof. It is not hard to check that $\widehat{N}(w_i;\Lambda_i) \subset N(0;\Lambda_i)$ for any cone Λ_i and any $w_i \in \Lambda_i$; see, e.g., [10, Proposition 2.1]. Then we have both conclusions of the corollary from Theorem 5.12. $\square$

Remark 5.14 (Comparison with known results)**.** For the case of finite set systems the intersection rules of Theorems 5.5 and 5.12 go back to the well-known results of [8]. In fact, not much has been known for representations of generalized normals to infinite intersections. Our previous results in this direction obtained in [10, 11], obtained on the base of the tangential extremal principle in finite dimensions, have a different nature and do not generally reduce to those in [8] for finite set systems.

An interesting representation of the basic normal cone (2.2) has been recently established in [15, Theorem 3.1] for infinite intersections of sets given by inequality constraints with smooth functions. This result essentially exploits specific features of the sets and functions under consideration and imposes certain assumptions, which are not required by our Theorem 5.12. In particular, [15, Theorem 3.1] requires the equicontinuity of the constraint functions involved, which is not the case of our Theorem 5.12 as shown in Examples 5.3 and 5.4. Note to this end that all the limiting normals are limiting R-normals in the framework of Example 5.3 and that the AQC assumption is satisfied therein; see Example 5.10.

We finish the paper with deriving necessary optimality conditions for problems of semi-infinite and infinite programming with geometric constraints given by

$$\text{minimize} \quad \varphi(x) \ \text{ subject to } x \in \emptyset_i, \quad t \in T, \tag{5.17}$$

with a general cost function $\varphi\colon X \to \overline{\mathbb{R}}$ and constraints sets $\emptyset_t \subset X$ indexed by an arbitrary (possibly infinite) set T. We refer the reader to [2, 4, 11] and the bibliographies therein for various results, discussions, and examples concerning optimization problems of type (5.17) and their specifications. The limiting normal cone representation (5.15) for infinite set intersections in Theorem 5.12, combined with some basic principles in constrained optimization, leads us to necessary optimality conditions for local optimal solutions to (5.17) expressed via its initial data.

The next theorem contains results of this kind in both *lower subdifferential* and *upper subdifferential* forms; see [9, Chapter 5] for general frameworks of constrained optimization and [2] for semi-infinite/infinite programs with linear inequality constraints in (5.17). The lower subdifferential condition is given below for the case of locally Lipschitzian cost functions on Asplund spaces via the construction

$$\partial\varphi(\bar{x}) := \mathop{\mathrm{Lim\,sup}}_{x\to\bar{x}} \widehat{\partial}\varphi(x)$$

known as the *Mordukhovich/basic/limiting subdifferential* of φ at $\bar{x}$; see [1, 8, 13, 14] for more details and discussions. The upper subdifferential condition below employs the so-called *Fréchet upper subdifferential/superdifferential* of φ at this point defined by

$$\widehat{\partial}^+\varphi(\bar{x}) := -\widehat{\partial}(-\varphi)(\bar{x}).$$

Theorem 5.15 (Necessary optimality condition for semi-infinite and infinite programs with general geometric constraints)**.** *Let $\bar{x}$ be a local optimal solution to problem* (5.17)*. Assume that any basic normal to $\emptyset := \bigcap_{i\in T} \emptyset_i$ at $\bar{x}$ is a limiting R-normal in this setting, and that the AQC requirements is satisfied for $\{\emptyset_i\}_{i\in T}$ at $\bar{x}$. Then the following conditions, involving finite index subsets $I \subset T$, hold:*

(i) *For general cost functions φ finite at $\bar{x}$ we have*

$$-\widehat{\partial}\varphi(\bar{x}) \subset \bigcap_{\varepsilon>0} \mathrm{cl}^*\Big\{ \sum_{i\in I} x_i^* + \varepsilon I\!B^* \;\Big|\; x_i^* \in \widehat{N}(x_i;\emptyset_i),\ \|x_i - \bar{x}\| < \varepsilon,\ I \subset T \Big\}. \tag{5.18}$$

(ii) *If in addition φ is locally Lipschitzian around $\bar{x}$, then*

$$0 \in \partial\varphi(\bar{x}) + \bigcap_{\varepsilon>0} \mathrm{cl}^*\Big\{ \sum_{i\in I} x_i^* + \varepsilon I\!B^* \;\Big|\; x_i^* \in \widehat{N}(x_i;\Omega_i),\ \|x_i - \bar{x}\| < \varepsilon,\ I \subset T \Big\}. \tag{5.19}$$

Proof. It follows from [9, Proposition 5.2] that

$$-\widehat{\partial}\varphi(\bar{x}) \subset \widehat{N}(\bar{x};\Omega) \subset N(\bar{x};\Omega) \tag{5.20}$$

for the general constrained optimization problem

$$\text{minimize } \ \varphi(x) \ \text{ subject to } \ x \in \Omega. \tag{5.21}$$

Employing now in (5.20) the intersection formula (5.15) for basic normals to $\Omega = \bigcap_{i\in T}\Omega_i$, we arrive at the upper subdifferential necessary optimality condition (5.18) for problem (5.17).

To justify (5.19), we get from [9, Propostion 5.3] the lower subdifferential necessary optimality condition

$$0 \in \partial\varphi(\bar{x}) + N(\bar{x};\Omega) \tag{5.22}$$

for problem (5.21) provided that φ is locally Lipschitzian around $\bar{x}$. Using the intersection formula (5.15) in (5.22) completes the proof of the theorem. $\square$

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